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On some subspaces for operators of class (N,k)

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Analisi funzionale. — *On some subspaces for operators of class* (N , k). Nota II di VASILE I. ISTRĂŢESCU, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Per gli operatori di classe (N , k) su spazi di Banach, cioè per operatori lineari e limitati su uno spazio di Banach aventi la proprietà che per ogni $x \in X$, $\|x\| = 1$ risulti $\|Tx\|^k \leq \|T^k x\|$, si dimostra che $m_T = \{x \in X, \|T^n x\| \leq \|x\|\}$ è un *sottospazio invariante* per tutti gli operatori che commutano con T. Vengono quindi studiate altre proprietà di tali operatori.

0. INTRODUCTION

In [2], an extension was given of a result of Lengyel and Stone for hermitian operators and of Halmos for normal operators [1].

The result established is as follows: if T is an operator of class (N), then the set

(*) $$m_T = \{x , x \in X , \|T^n x\| \leq \|x\|\}$$

is a closed linear subspace which is invariant for all operators commuting with T. The purpose of the present Note is to further extend the above result, proving in fact that m_T is an invariant subspace for all operators commuting with T, if we suppose that T is of class (N , k), i.e., for all $x \in X$ (X being a Banach space) we have that

$$\|Tx\|^k \leq \|T^k x\| \, \|x\|^{k-1}$$

[when $k = 2$, T is called of class (N)]. Also we give a result about operators induced on a quotient space when T is of class (N) or class (N , k).

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1. SUBSPACES FOR OPERATORS OF CLASS (N , k)

Let X be a Banach space and T a bounded linear operator on X.

DEFINITION 1.1. We say that T is of class (N , k) if for all $x \in X$, $\|x\| = 1$

$$\|Tx\|^k \leq \|T^k x\|$$

[when $k = 2$ we say that T is of class (N)].

(*) Nella seduta del 10 giugno 1976.

Our result is the following

THEOREM 1.2. *If* T *is of class* (N , k), *then the set defined in* (*) *is a closed linear subspace invariant for all operators commuting with* T.

Proof. First we consider the set

$$\tilde{m}_{\mathrm{T}} = \{x\,,\{\|\mathrm{T}^n x\| \leq \|x\|\} \text{ is a bounded sequence}\}$$

and it is obvious that $\tilde{m}_{\mathrm{T}}$ is a linear subspace invariant for all operators commuting with T. To prove the theorem it remains to show that $\tilde{m}_{\mathrm{T}}$ is closed and $m_{\mathrm{T}} = \tilde{m}_{\mathrm{T}}$. Since $m_{\mathrm{T}} \subset \tilde{m}_{\mathrm{T}}$ we suppose, on the contrary, that there exists $x_0 \in \tilde{m}_{\mathrm{T}}$ and $x_0 \notin m_{\mathrm{T}}$. We can suppose, without loss of generality, that $\|\mathrm{T}\| = 1$. Let n_0 be the first integer such that $\|x_0\| < \|\mathrm{T}^{n_0} x_0\|$ and we remark that we can suppose that $\|x_0\| > 1$. Let $1 < \alpha = \|\mathrm{T}^{n_0} x_0\|$. Since T is of class (N , k) we have

$$\|\mathrm{T}^{n_0} x_0\|^k = \|\mathrm{T}(\mathrm{T}^{n_0-1} x_0)\|^k =$$

$$= \left\| \mathrm{T} \frac{\mathrm{T}^{n_0-1} x_0}{\|\mathrm{T}^{n_0-1} x_0\|} \right\|^k \|\mathrm{T}^{n_0-1} x\|^k \leq \|\mathrm{T}^{n_0+k-1} x_0\| \, \|\mathrm{T}^{n_0-1} x_0\|^{k-1}$$

and thus, by the definition of n_0,

$$(**) \qquad \|\mathrm{T}^{n_0+k-1} x_0\| \geq (\|\mathrm{T}^{n_0} x_0\|^k)\,.$$

Now we have further

$$\|\mathrm{T}^{n_0+k-1} x_0\| \leq \|\mathrm{T}^{n_0} x_0\| \, \|\mathrm{T}^{k-2}\| \leq \|\mathrm{T}^{n_0} x_0\|$$

and, since

$$\|\mathrm{T}^{n_0+2k-2} x_0\| = \left\| \mathrm{T}^k \frac{\mathrm{T}^{n_0+k-2} x_0}{\|\mathrm{T}^{n_0+k-2} x_0\|} \right\| \|\mathrm{T}^{n_0+k-2} x_0\| \geq \|\mathrm{T}^{n_0+k-1} x_0\|\,,$$

we obtain the inequality

$$(***) \qquad \|\mathrm{T}^{n_0+2k-2} x_0\| \geq \|\mathrm{T}^{n_0+k-1} x_0\|^k \, \|\mathrm{T}^{n_0} x_0\|^{1-k} \geq \|\mathrm{T}^{n_0} x_0\|^{k^2-k+1}\,;$$

moreover, for all l and j we have the inequality (we apply the fact that T is of class (N , k)), $(0 < j < k - 1)$,

$$\|\mathrm{T}^{n_0+lk-j} x_0\| = \left\| \mathrm{T}^k \frac{\mathrm{T}^{n_0+(l-1)k+j} x_0}{\|\mathrm{T}^{n_0+(l-1)k+j} x_0\|} \right\| \|\mathrm{T}^{n_0+(l-1)k+j} x_0\| \geq$$

$$\geq \|\mathrm{T}^{n_0+(l-1)k+j+1} x_0\|^k \, \|\mathrm{T}^{n_0+(l-1)k+j}\|^{1-k} \geq \|\mathrm{T}^{n_0+(l-1)k+j+1} x_0\| \, \|\mathrm{T}^{n_0} x_0\|^{1-k}.$$

From this inequality and (** , (***) we find a sequence of integers (m_k) such that $\{\| T^{mk} x_0 \|\}$ is not bounded and this represents a contradiction. Thus $\tilde{m}_T \subset m_T$ and since m_T is obviously closed the theorem is proved.

Remark 1.3. We recall that it is not known whether the following property holds for operators of class (N , k): if T is of class (N , k), then is it true that any power of T is of class (N , k)? If such a property holds, then an easy proof of Theorem 1.2 can be deduced.

2. INDUCED OPERATORS ON QUOTIENT SPACES

Suppose now that X is a Banach space and let T be a bounded linear operator on X with the subspace X_1 as an invariant subspace. We can define the quotient subspace $\tilde{X}_1$ as all equivalence classes $[x]$ with the norm

$$\| [x] \| = \inf_{x \in [x]} \| x \|$$

and it is known that $\tilde{X}_1$ becomes a Banach space. Since X_1, is invariant under T, we can define a operator T_1 on $\tilde{X}_1$ as follows

$$T_1 [x] = [Tx]$$

and it is easy to see that $\| T_1 \| \leq \| T \|$. We can prove the following

THEOREM 2.1. *If* T *is of class* (N , k) *then* T_1 *is of class* (N , k).

Proof. We give the proof only in the case of operators of class (N) since the case $k \neq 2$ does not differ essentially from $k = 2$.

Let $[x] \in \tilde{X}_1$ and thus

$$T_1 [x] = [Tx] .$$

We have

$$\| T_1^2 [x] \| = \| [T^2 x] \| = \inf_{x \in [x]} \| T^2 x \|$$

and, since T is of class (N), we have for all $z \in [x]$

$$\| T^2 z \| \; \| z \| \geq \| Tz \|^2.$$

Let $\{z_n\} \in [x]$ such that

$$\| T_1^2 [x] \| = \lim \| T^2 z_n \| ;$$

thus, if $\| [x] \| = 1$,

$$\| T_1^2 [x] \| = \lim \| T^2 z_n \| \geq \lim \inf \| Tz_n \|^2 \geq \| [Tz_n] \|^2$$

and the theorem is proved.

COROLLARY 2.2. *If* T *is of class* (N), *and* X_1 *is an invariant subspace for* T *with the property that* T^{-1} *exists and* $T^{-1}\mathfrak{X}_1 \subset X_1$, *then* T_1^{-1} *is of class* (N).

Proof. Since T_1^{-1} is bounded, the corollary follows from Theorem 2.1.28 of [3].

REFERENCES

[1] P. HALMOS (1951) – *Introduction to Hilbert spaces*, Chelsea Publ. C., N. Y.

[2] V. ISTRĂŢESCU (1967) – *On some subspaces of operators of class* (N), « Rend. Accad. Naz. dei Lincei », ser. VIII, *48*, 305–306.

[3] V. ISTRĂŢESCU – *Topics in operator theory. Lectures notes on operator theory*, in « Contributi del Centro Linceo Interdisciplinare di Scienze Matematiche e loro Applicazioni (to appear).