Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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On a class of submodules in direct products

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **60** (1976), n.6, p. 743–748.

Accademia Nazionale dei Lincei

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Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1976.

Algebra. — *On a class of submodules in direct products.* Nota di LASZLO FUCHS e FRANS LOONSTRA, presentata (*) dal Corrisp. G. ZAPPA.

RIASSUNTO. — In una somma diretta finita di moduli, i sottomoduli possono costruirsi usando certi omomorfismi e certe equazioni. Nel presente lavoro si studiano quei sottomoduli che possono ottenersi in modo simile in un prodotto infinito di moduli.

In spite of the widely recognized importance of subdirect products of modules (or groups, rings etc.), not much is known about their structures in general. An exception is the case of two modules when subdirect products are pullbacks of epimorphisms. Subdirect products of many modules which can be obtained essentially in the same way have been investigated in [2].

For more than two modules, no satisfactory description of subdirect products is available. In the finite case, however, somewhat more can be said. Observe that if $M_1, \cdots, M_k$ is a finite set of modules and M is any submodule of their direct sum $M^* = M_1 \oplus \cdots \oplus M_k$ (not necessarily a subdirect sum), then we have, for each i, a homomorphism

$$\alpha_i : M_i \to F = M^*/M \qquad (m_i \mapsto m_i + M),$$

such that $(m_1, \cdots, m_k) \in M^*$ belongs to M exactly if $\alpha_1 m_1 + \cdots + \alpha_k m_k = 0$. We interpret this as saying that the submodules of M^* can be singled out with the aid of homomorphisms $\alpha_i : M_i \to F$ and equations like $\alpha_1 x_1 + \cdots + \alpha_k x_k = 0$. This is not true in general in the infinite case, and we set ourselves the goal of studying those submodules in infinite products of modules that can be constructed in a similar fashion via homomorphisms and equations.

After establishing a universal property of the class of submodules under consideration, we give a topological characterization, pointing out an interesting connection between algebraic and topological aspects. We shall see that exactly the linearly complete modules can be represented non-trivially in the indicated way.

§ 1. Let R denote an arbitrary ring with 1. All modules will be unital left R-modules and all maps between them will be R-homomorphisms.

We study modules M defined in terms of two sets $\{M_i\}_{i \in I}$ and $\{F_j\}_{j \in J}$ of modules and a system $\{\alpha_{ji}\}$ of homomorphisms

$$\alpha_{ji} : M_i \to F_j \qquad \text{(for all } i \in I, j \in J) \tag{1}$$

(*) Nella seduta del 10 giugno 1976.

where, for each $j \in J$, almost all α_{ji} are zero. In addition, assume that we are given a "system of equations"

$$\sum_{i \in I} \alpha_{ji} x_i = 0 \qquad \text{(for all } j \in J) . \tag{2}$$

We then define the module M as the submodule of the product $M^* = \prod_{i \in I} M_i$ consisting of all $m^* = (\cdots, m_i, \cdots) \in M^*$ $(m_i \in M_i)$ which satisfy

$$\sum_{i \in I} \alpha_{ji} m_i = 0 \qquad \text{(for all } j \in J) . \tag{3}$$

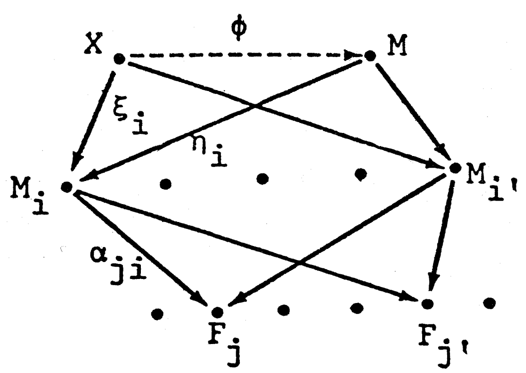
We shall call this M a *general subproduct* of the M_i (it need not be a subdirect product).

For every $j \in J$, there is a homomorphism $\alpha_j : M^* \to F_j$ such that if $\rho_i : M_i \to M^*$ are the injection maps, then $\alpha_j \rho_i = \alpha_{ji}$ for every i. In fact, $\alpha_j (\cdots, m_i, \cdots) = \Sigma \alpha_{ji} m_i$ is such a map.

Note that an $m^* = (\cdots, m_j, \cdots) \in M^*$ belongs to M exactly if $m^* \in \operatorname{Ker} \alpha_j = K_j$ for every j, i.e.

$$M = \bigcap_{j \in J} K_j .$$

Given $j \in J$, let $\{i_1, \cdots, i_k\}$ be the set of those $i \in I$ for which $\alpha_{ji} \neq 0$ and write $N_j = M_{i_1} \oplus \cdots \oplus M_{i_k}$. The α_{ji} induce a homomorphism $\beta_j : N_j \to F_j$ such that α_j factors through the obvious projection $\gamma_j : M^* \to N_j$ as follows: $\alpha_j = \beta_j \gamma_j$. Consequently, the F_j can be chosen as quotients of finite direct sums of the M_j's.



Let us point out a universal property of general subproducts. Suppose we are given the modules $\{M_i\}_{i \in I}$, $\{F_j\}_{j \in J}$, the homomorphisms (1) and the equations (2). By a *solution* of (2) we mean a module X together with a set of homomorphisms $\xi_i : X \to M_i$ $(i \in I)$ such that (2) holds when the x_i are replaced by the ξ_i. By a *universal solution* we mean a solution $\eta_i : M \to M_i$ $(i \in I)$ such that for any solution X there is a unique homomorphism $\varphi : X \to M$ with $\eta_i \varphi = \xi_i$ for every i.

THEOREM 1. *The module* M, *defined above as a general subproduct, is a universal solution of* (2). *Every universal solution of* (2) *is isomorphic to* M.

In fact, the restriction of the i-th projection $\Pi M_i \to M_i$ to M is a solution $\eta_i : M \to M_i$, and if $\xi_i : X \to M_i (i \in I)$ is any solution, then $\varphi : x \mapsto (\cdots, \xi_i x, \cdots)$ is a map $\varphi : X \to M$ as required. That M is up to isomorphism the only universal solution follows in the standard way.

It should be pointed out that inverse limits can also be regarded as general subproducts. If $\{M_i ; \pi_i^k (i \leq k)\}$ is an inverse system of modules, then define $F_j = M_i$ for all $j = \{i, k\}$ with $i \leqq k$, and $\alpha_{ji} = -\pi_i^i$, $\alpha_{jk} = \pi_i^k$ and $\alpha_{jl} = 0$ otherwise. It is readily seen that the corresponding general subproduct is nothing else than the inverse limit of the system. In § 3 we shall see that a general subproduct can be obtained as the limit of an appropriately chosen inverse system.

§ 2. Our next objective is to find a characterization of those submodules of the direct product $M^* = \Pi M_i$ of the modules M_i that appear as general subproducts for suitable choices of F_j and α_{ji}.

Consider the modules M_i as being equipped with the discrete topologies and furnish M^* with the product topology. This gives rise to a linear Hausdorff topology on M^* in which it is complete (we shall indicate this situation by saying: it is *linearly complete*). If not stated otherwise, M^* is assumed to carry this topology. Manifestly, the maps α_j are continuous (let the F_j be again discrete). Furthermore, all the K_j are open submodules, since each contains the product of almost all M_i. Consequently, they are closed in M^*, and so is their intersection M. It follows that M is linearly complete in the topology inherited from M^*. This establishes the necessary assertion of

THEOREM 2. *A submodule of* $M^* = \Pi M_i$ *is a general subproduct of the* M_i *exactly if it is a closed submodule of* M^*.

To verify sufficiency, suppose M is a closed submodule of M^*. Define the index set J to consist of all finite subsets $\{i_1, \cdots, i_k\} = j$ of I, and set $N_j = M_{i_1} \oplus \cdots \oplus M_{i_k}$ with $\gamma_j : M^* \to N_j$ as projection map. For every $j \in J$, we set

$$K_j = \gamma_j^{-1}(\gamma_j M) \qquad \text{and} \quad F_j = M^*/K_j, \tag{4}$$

and define $\alpha_{ji} : M_i \to F_j$ via $\alpha_{ji} m_i = \rho_i m_i + K_j$ where $\rho_i : M_i \to M^*$ is the injection map. Clearly, $\alpha_{ji} = 0$ for all $i \notin j$, and $m^* = (\cdots, m_i, \cdots) \in M^*$ satisfies $\sum_i \alpha_{ji} m_i = 0$ exactly if $m^* \in K_j$. We see that the general subproduct of the M_i defined by the equations $\Sigma \alpha_{ji} x_i = 0$ for all $j \in J$ is precisely the intersection of all K_j in (4). Evidently, M is contained in $\bigcap_j K_j$. On the other hand, $m^* \in K_j$ means that, for some $m \in M$, $\gamma_j m^* = \gamma_j m$, i.e., $m^* - m \in \operatorname{Ker} \gamma_j = \prod_{i \notin j} M_i$. But these products, taken for all j, form a base of open neighbourhoods about zero in M^*, so that every $m^* \in \cap K_j$ is in the closure of M in M^*. Since M is closed, $M = \cap K_j$ is a general subproduct, indeed.

The last result can be extended to the case when the modules occurring carry some (Hausdorff) topologies. Naturally, we then assume all maps α_{ji} in (1) continuous. If we furnish $M^* = \Pi M_i$ again with the product topology, then the α_j are continuous, so Ker α_j are closed in M^*, and so is their intersection M. Moreover, the proof of Theorem 2 can readily be extended to the case of topological modules to obtain:

THEOREM 2a. *Let* $M^* = \Pi M_i$ *be the product of topological modules* M_i. *A submodule of* M^* *is a general subproduct exactly if it is a closed submodule of* M^*.

In particular, the compact submodules of a product of compact modules are precisely the general subproducts.

§ 3. We turn our attention to the question as to when a module M can be represented as a general subproduct. Manifestly, to avoid trivialities, we require not only an infinite number of components, but consider only general subproducts such that a projection to the direct sum of finitely many components is never a monomorphism.

THEOREM 3. *A module* M *can be represented in a non-trivial fashion as a general subproduct if and only if* M *is Hausdorff linearly complete in some non-discrete topology.*

By what has been said before Theorem 2, it is evident that every general subproduct is linearly complete in the topology inherited from M^*. If M is a non-trivial subproduct, then its intersection with open submodules of M^* is never zero. Conversely, let M be a non-discrete, linearly complete module, and $\{U_i\}_{i \in I}$ a subbase of open neighbourhoods about zero, all submodules of M. Since the topology is Hausdorff, $\bigcap_i U_i = 0$ is clear. Consequently, the natural maps of M onto the quotients $M/U_i = M_i$ induce an embedding $\varphi : M \to \Pi M_i = M^*$ (where again M^* is awarded the product topology) such that φ is a topological isomorphism of M with its image. By completeness, φM is closed in M^*, so a simple appeal to Theorem 2 convinces us that M is a general subproduct. It is a non-trivial one, because the intersection of any finite number of the U_i is not 0.

De Marco and Orsatti have shown [1] that an abelian group admits a non-discrete linearly complete topology if and only if it contains a subgroup which is a direct sum of infinitely many non-zero cyclic groups, i.e. it is of infinite rank. Consequently, *an abelian group can be represented non-trivially as a general subproduct exactly if it is of infinite rank.*

For arbitrary modules, the existence of non-discrete linearly complete topologies is an open problem. From [1] it follows that a module of infinite Goldie dimension does admit such a topology. But it can happen that a module of rank 1 has such a topology as is shown by the free module of rank 1 over the p-adic integers (in the p-adic topology).

Linearly complete modules are precisely the inverse limits of discrete modules, thus Theorem 3 implies that general sub-products can be obtained as inverse limits. (An inverse system can be constructed in a way described in the proof of this theorem).

§ 4. In view of the last remark, it is not at all surprising that Hom commutes with general subproducts in the second variable:

THEOREM 4. *Let* M *be the subproduct of the modules* M_i *which is defined in terms of* (1) *and* (2). *For any* R-*module* N, $\mathrm{Hom}_R(N, M)$ *is, as an abelian group, the subproduct defined via the homomorphisms*

$$\mathrm{Hom}_R(1_N, \alpha_{ji}) : \mathrm{Hom}_R(N, M_i) \to \mathrm{Hom}_R(N, F_j)$$

and the equations

$$\sum_i \mathrm{Hom}_R(1_N, \alpha_{ji})\, x_i = 0 \qquad \textit{for all } j.$$

The proof is straightforward and may be left to the reader.

As the behavior of Ext is different, it is not expected to commute with general subproducts, with the exception of very special cases.

§ 5. It is natural to inquire about the structure of general subproducts if the structures of the M_i are known. In general, not much can be said, but in certain particular cases more information can be obtained.

Here we concentrate on the case when all the M_i are injective. Recall that the singular submodule $Z_R(N)$ of a module N is defined as the set of all $x \in N$ such that

$$\mathrm{Ann}\, x = \{r \in R \mid rx = 0\} \quad \text{is an essential left ideal of R.}$$

THEOREM 5. *If* $M_i\ (i \in I)$ *are all injective and if* $Z_R(F_j) = 0$ *for every* j, *then a subproduct* M *defined in terms of* (1) *and* (2) *is again injective.*

In view of the injectivity of $M^* = \Pi M_i$, the proof can be restricted to verifying the following: given $\varphi : L \to M$ (where L is a left ideal of R), let $\psi : R \to M^*$ be an extension of φ; then $\psi(1) \in M$ for the identity 1 of R. It suffices to do this for essential left ideals L of R. Set $\psi(1) = (m_1, \cdots, m_i, \cdots) \in M^*$. For every $r \in L$, $(rm_1, \cdots, rm_i, \cdots) = r\psi(1) = \varphi(r) \in M$, thus

$$r\left(\sum_i \alpha_{ji} m_i\right) = \sum_i \alpha_{ji}(rm_i) = 0 \qquad \text{for every } j$$

and for every $r \in L$. Therefore

$$\mathrm{Ann}\left(\sum_i \alpha_{ji} m_i\right) \geqq L,$$

whence $\sum_i \alpha_{ji} m_i \in Z(F_j) = 0$ follows, for every j. Consequently, $\psi(1) \in M$, q.e.d.

If R is a commutative domain, a similar result can be established with "divisible" replacing "injectivity"; in this case the F_j are supposed to be torsion-free.

References

[1] G. De Marco and A. Orsatti (1976) – *Complete linear topologies on abelian groups*, « Symposia Math. », *13*, 153–161.

[2] F. Loonstra (1962) – *Über subdirekte Produkte von Gruppen*, « Rend. Mat. e Appl. », (5) *21*, 364–372.