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Some Stability Problems from a Topological Viewpoint

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SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *Some Stability Problems from a Topological Viewpoint.* Nota di FEDERICO MARCHETTI, presentata (*) dal Socio C. CATTANEO.

RIASSUNTO. — In questo lavoro viene studiato il concetto di stabilità totale, per sistemi dinamici discreti, attraverso gli ampliamenti conseguenti alla introduzione di convenienti topologie. In questo ambito è risultato naturale definire ed analizzare un concetto di stabilità totale in un senso più forte di quello usuale. Infine i risultati sono stati applicati alla teoria della biforcazione.

0. INTRODUCTION

In this paper we discuss some questions relating to stability problems for dynamical systems subjected to " persistent perturbations ", that is total stability. We propose a rather general definition and carry out some developements toward a characterization of these properties using systematically a topological approach. This procedure offers, in our opinion, a number of advantages. It gives indeed, in a natural way, a rather general frame in which various kinds of gauging of perturbations on a dynamical systems can be unified. Moreover proofs tend to be simplified and easily generalizable to related problems.

This simplicity is obviously due to the fact that, since all stability properties are essentially topological, a method bringing this character in the forefront should rapidly focus on the essential structural aspects of the questions. It should be remarked that the standard method of investigation, Lyapunov functions, is basically in this line. It is well-known that the standard

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properties of Lyapunov functions, e.g. in stability or asymptotic stability, are equivalent to a definite behaviour of a base of neighbourhoods under the action of the dynamical system (positive invariance, monotone shrinking to the equilibrium point, etc.). The link is obviously provided, in metrizable spaces, by the Urysohn functions associated to the equilibrium point and the complement of each of these neighnourhoods. The use of Lyapunov functions is a powerful tool in applications, since it gives an analytical tool, which in a concrete problem may actually be computed. On the other hand it may happen that, as far as fundamentals are concerned, it could be simpler to investigate directly the neighbourhoods. Moreover some generality is thus gained, since we are no longer bound to metrizable spaces. For instance, in a T_3-space, which is not 1^{st} countable, the Urysohn functions cited above may still be considered, but it will be in general impossible to build a single Lyapunov function (or finetely or countably many of such) out of them. We are thus naturally led to recover a concept introduced by Salvadori in [1], where a continuous family of "Lyapunov" functions was introduced as a more flexible tool. We also recall that in [2] some preliminary discussion on relations between existence of appropriate functions and dynamical behaviour of bases of neighbourhoods is carried out.

In [3] and [4] Lyapunov functions (specifically associated to asymptotic stability) prove to be extremely helpful in problems of total stability. Our purpose is to give some new results and generalizations concentrating on the behaviour of certain bases of neighbourhoods. We shall consider in this paper discrete systems only. Further generalizations and questions will be studied in a subsequent paper. The point is that discrete systems are geometrically simpler and help to simplify the approach we propose to develop. As far as these systems are concerned, the advantages envisaged above should result from the following.

1. Notations

We shall consider discrete systems only. That is we consider a continuous map φ of a topological space X in itself, with all its positive powers. We denote by C (X) the set of all continuous functions of X in itself, which is a semi-group under composition.

If $A \subset X$, $\bar{A}$, $\mathring{A}$, ∂A , A^c denote closure, interior, boundary and complement of A respectively. For any topological space Y, $y \in Y$, $\mathcal{N}_Y(y)$ or $\mathcal{N}(y)$ if no confusion is likely, denotes the filter of neighbourhoods of y in Y.

If R and S are expressions, R : = S indicates a defining relation for R.

1.1. Definition. $A \subset X$ is said to be

1.1.1. Positively invariant if $\varphi A \subset A$.

1.1.2. Invariant if $\varphi A = A$.

1.1.3. Contracting if $\varphi^{-1} \mathring{A} \supset \bar{A}$.

If X is a metric space with metric d, we shall call A

1.1.4. Strictly contracting if $d(\varphi\bar{A}, \overline{A^c}) > 0$.

If more than one system is under consideration we shall also speak of a set as φ–invariant, φ–contracting etc.

2. TOTAL STABILITY

We want to define a general concept of total stability for the discrete systems we are considering. For this assume that we are considering systems generated by the elements of a certain subsemigroup of C (X), G (X) (which will be dictated by the problem at hand). Let $\mathcal{C}$ be a topology on $G(X) \times X$. Projection on the factors naturally induces a topology on them, and we denote by $\mathcal{G}$ and $\mathcal{X}$ the resulting spaces. In general $\mathcal{X}$ will be different from X, though the underlying sets will be equal.

2.1. DEFINITION.

2.1.1. $(\varphi, x_0) \in G(X) \times X$ is $\mathcal{C}$-totally stable if

$$\forall \mathcal{U} \in \mathcal{N}_X(x_0)\ \exists \mathcal{W} \in \mathcal{N}_{G(X)\times X}(\varphi, x_0) : (\psi, x) \in \mathcal{W} \Rightarrow \psi^n x \in \mathcal{U} \quad n = 1, 2, \cdots$$

2.1.2. $(\varphi, x_0) \in G(X) \times X$ is strongly $\mathcal{C}$-totally stable if

$$\forall \mathcal{U} \in \mathcal{N}_X(x_0)\ \exists \mathcal{V} \in \mathcal{N}_{\mathcal{X}}(x_0)\ , \quad \exists \mathcal{W} \in \mathcal{N}_{\mathcal{G}}(\varphi) : \mathcal{W}^n \mathcal{V} \subset \mathcal{U} \quad n = 1, 2, \cdots$$

(Here exponentiation of a set is meant in the algebraic sense).

2.2. *Remark.*

2.2.1. The generalization of the above definitions to (φ, C) where C is a (generally closed or compact) subset of X, is obvious and routine.

2.2.2. If X is T_1 (separability of $\mathcal{X}$ is immaterial), for any choice of $\mathcal{C}$, total stability of (φ, x_0) implies $\varphi x_0 = x_0$.

2.2.3. If $\mathcal{C}$ is such that $\mathcal{X}$ is weaker than X (i.e. the identity map $id : X \to \mathcal{X}$ is continuous) $\mathcal{C}$-total stability implies φ-stability in the sense of Lyapunov.

2.2.4. Total stability implies continuity of the evaluation map:

$$ev : G(X) \times X \to X \quad , \quad ev : (\varphi, x) \mapsto \varphi x \qquad \text{at} \quad (\varphi, x_0).$$

2.2.5. In this frame it is natural to introduce different topologies on the same set. This suggests that these kind of properties find their natural setting in a more general frame than the standard theory for dynamical systems. In this line one should compare this with a theory such as developed in [5].

2.2.6. In an intuitively clear sense strong total stability is stability under perturbations which explicitly depend on 'time'. In [6] ordinary autonomous differential equations are perturbed in much the same sense (i.e. if we consider a discrete system generated by ' discretizing ' time, this system is strongly $\mathscr{T}$-totally stable, by a suitable choice of $\mathscr{T}$, if the original one is totally stable in the sense of Seibert).

To illustrate further the definition of strong total stability consider the following simple example.

Assume we have an infinite transmission line, i.e. a sequence of maps (" black boxes ") mapping a certain space (whose elements are the " signals ") in itself. Suppose the " ideal " line to be such that all " boxes " are equal and admit a certain element of the space as a fixed point (" the signal should travel undisturbed along the line "). Suppose that a certain topology has been chosen on the space of signals, such that the maps are continuous. Strong total stability describes the (usually desirable) property that a small error in the realization of the boxes and of the signal (" small " in a sense specified by the choice of $\mathscr{T}$) will result in a small change in the resulting signal, all along the line.

Different choices of $\mathscr{T}$ will result in different properties, of course (possibly trivial ones: for instance if $\mathscr{T}$ is discrete any couple $(\varphi, x) : \varphi x = x$ is strongly totally stable). It is a natural problem to try and classify various types of stability that have been introduced in terms of topologies. Here we shall study some possible choices, which appear rather natural, and investigate some sufficient and (in particular but significant cases) necessary conditions.

The simplest choice is to choose a topology on G(X) and choose $\mathscr{T}$ as a product topology, since X already is endowed with one. In this respect it should be noted that, if X is locally compact, we have a natural choice at hand, namely the compact-open topology on C(X), which is the weakest which ensures continuity of the evaluation map (cfr. [7]). On the other hand one may wish to choose a weaker topology, choosing as $\mathscr{X}$ a space where the open sets are the open sets in X, which contain the point whose stability we are considering. Again, if X (and hence $\mathscr{X}$) are locally compact, the compact-open topology (from $\mathscr{X}$ to X) will be a natural choice. We shall also consider the choices made in [3] and [4] (the latter for metrizable spaces).

If the space X is not locally compact, it is well-known that the compact-open topology loses its advantages. However most results may be extended by adapting the preceding definitions, substituting " closed " for " compact ". A particular interesting case is present if X has a normed linear structure and G(X) is the space of Lipschitz continuous functions.

Considerations above should help and motivate the following definitions.

2.3. DEFINITION.

Let X be locally compact and $x_0 \in X$.

2.3.1. We denote by $\mathscr{K}_0$ the product topology on $G(X) \times X$ induced by the compact open topology on G(X) and the natural one on X.

2.3.2. We denote by $\mathcal{K}_1(x_0)$ the product topology on $G(X)\times\mathcal{X}$ induced by the compact-open topology (from $\mathcal{X}$ to X) on G (X) and the natural one on $\mathcal{X}$, where the open sets in $\mathcal{X}$ are the open sets in X which contain x_0.

2.3.3. We denote by $\mathcal{K}_2(x_0)$ the topology on $G(X)\times X$, a subbase of which is given by sets of the form $\bigcup_{\alpha\in A} \mathcal{U}_\alpha\times\mathcal{V}_\alpha$, where $\{\bar{\mathcal{V}}_\alpha\}_{\alpha\in A}$ is a base of precompact neighbourhoods of x_0 and

$$\mathcal{U}_\alpha := \{\varphi\in G(X) : \varphi(\bar{\mathcal{V}}_\alpha)\subset G_\alpha, G_\alpha \text{ open in } X\}.$$

Let X be not locally compact and $x_0\in X$

2.3.4. We denote by $\mathcal{C}_0$ the topology on $G(X)\times X$ defined as in 2.3.1 except that closed sets are considered instead of compact ones.

2.3.5. We denote by $\mathcal{C}_1(x_0)$ the topology on $G(X)\times X$ defined as in 2.3.2 except that closed sets are considered instead of compact ones.

2.3.6. We denote by $\mathcal{C}_2(x_0)$ the topology on $G(X)\times X$ defined as in 2.3.3 except that closed sets are considered instead of compact ones.

Let X be metrizable and choose a metric ρ (e.g. bounded).

2.3.7. We denote by $\mathcal{D}_0$ the product topology on $G(X)\times X$ induced by the natural topology on X and the following metric on G (X)

$$\rho_G(\varphi, \psi) := \sup_{x\in X} \rho(\varphi x, \psi x).$$

2.3.8. We denote by $\mathcal{L}_0$ the product topology on $\mathrm{Lip}(X)\times X$, where Lip (X) is the space of Lipschitz functions on X, induced by the Lipschitz metric on Lip (X) and the natural topology on X.

2.4. *Remark.*

2.4.1. If X is a metric space, $\mathcal{K}_2(x)$-total stability coincides with total stability as defined in [3] (we are considering the discrete case only); $\mathcal{D}_0$-total stability was introduced in [4]. There is an obvious order of weakness in the topologies defined above.

2.4.2. Motivation for definitions like those where one does not adopt a product topology on $G(X)\times X$, is the consideration of the local character of stability, so that it seems natural to let only the neighbourhoords of the particular point x_0 under scrutiny play a rôle.

Of course if X has an underlying topological group structure, translations relate all open sets and these distinctions become void.

2.4.3. Note that in case $\mathcal{K}_0, \mathcal{C}_0, \mathcal{D}_0, \mathcal{L}_0$ the identity map $id:\mathcal{X}\to X$ is continuous, whereas in the other cases it will be continuous only at x_0; it will always be open.

2.4.4. The rather peculiar character of Definitions 2.3.3 and 2.3.6 is motivated by considering that a stability condition, which prevents the evolution of the system from leaving a neighbourhood, when starting from another one, justifies matching two systems over the relevant neighbourhood only. An uncomfortable side effect is the much weaker form which the following theorem assumes in this case.

2.5. THEOREM.

2.5.1. *If* x_0 *admits a* φ_0*-contracting base of neighbourhoods,* (φ_0, x_0) *is strongly* $\mathfrak{T}$*-totally stable, for* $\mathfrak{T} = \mathscr{K}_0$, $\mathfrak{T} = \mathscr{K}_1(x_0)$ *or* $\mathfrak{T} = \mathscr{C}_0$, $\mathfrak{T} = \mathscr{C}_1(x_0)$ *as the case may be.*

2.5.2. *If* x_0 *admits a* φ_0*-contracting base of neighbourhoods,* (φ_0, x_0) *is* $\mathscr{K}_2(x_0)$*-totally stable (or* $\mathscr{C}_2(x_0)$*-totally stable, as the case may be), though in general not* $\mathscr{K}_2(x_0)$ *(or* $\mathscr{C}_2(x_0)$*)-strongly totally stable.*

Proof. 2.5.1 is straightforward, since by definition, for each contracting neighbourhood $\mathscr{V}$ of x_0 there is a corresponding neighbourhood of $\varphi_0 : \mathscr{W}_{\mathscr{V}} := \{\psi \in \mathrm{G}(\mathrm{X}) : \psi(\overline{\mathscr{V}}) \subset \mathscr{V}\}$ and it is obvious that, given $\mathscr{U} \in \mathscr{N}_{\mathrm{X}}(x_0)$, chosen a contracting neighbourhood $\mathscr{V} \subset \mathscr{U}$, $\mathscr{V}$ and $\mathscr{W}_{\mathscr{V}}$ satisfy Definition 2.1.2. As for 2.5.2 build a neighbourhood of (φ_0, x_0) using the φ_0-contracting base of neighbourhoods of x_0:

$$\alpha \in \mathrm{A} \qquad \mathscr{U}_\alpha = \{\psi \in \mathrm{G}(\mathrm{X}) : \psi(\overline{\mathscr{V}_\alpha}) \subset \mathscr{V}_\alpha\} \quad \text{and put} \quad \mathscr{W} := \bigcup_{\alpha \in \mathrm{A}} \mathscr{U}_\alpha \times \mathscr{V}_\alpha .$$

If the base is contained in the given $\mathscr{U} \in \mathscr{N}_{\mathrm{X}}(x_0)$, Definition 2.1.1 is obviously satisfied.

2.6. COROLLARY. *If* X *is metrizable (and* φ_0 *is Lipschitz continuous), if* x_0 *admits a* φ_0*-contracting base of neighbourhoods,* (φ_0, x_0) *is strongly* $\mathscr{D}_0$*-totally stable (*$\mathscr{L}_0$*-totally stable).*

Proof. Immediate by comparing with the topologies $\mathscr{K}_0$ or $\mathscr{C}_0$ as the case may be.

2.7. *Remark.*

2.7.1. Corollary 2.6 holds in particular if the neighbourhoods considered are strictly contracting (which is the same as contracting in the locally compact case); in this case it is easy to check the corollary directly as well as the theorem.

2.7.2. Assume that $\overline{\mathscr{V}} \neq \varphi_0^{-1}\mathscr{V}$, i.e. that contractivity is effectively stronger than positive invariance, and that X is perfectly normal [7]; the Urysohn function associated to x_0 and $\overline{\mathscr{V}^c}$ will be strictly φ_0-decreasing on $\partial\mathscr{V}$, and will stay so for a " small " perturbation of φ_0; considerations such as these should provide a link with Lyapunov function methods (cfr. the introduction).

Theorem 2.5.2 reduces to the analogous theorem in [3] for discrete systems (if X is metric locally compact), because of Remark II, 3.2 there and the following easy lemma.

2.8. LEMMA. *If* X *is a topological space (is a metric locally compact space). existence of a* φ*-contracting base of neighbourhoods of* x *implies (is equivalent to) existence of a base of uniformly* φ*-asymptotically stable neighbourhoods of* x.

Proof. If $\mathscr{V}$ is a contracting neighbourhood, $\overline{\mathscr{V}}$ attracts uniformly its neighbourhood $\varphi^{-1}\mathscr{V}$; in a metric locally compact space, a compact uniformly asymptotically stable set can be associated to a Lyapunov function, continuous, positive off the stable set and strictly decreasing along the trajectories attracted to the set (see e.g. [8]): it is easy to prove the inverse part of the lemma with the help of these functions.

3. A CONVERSE THEOREM

To try and find a necessary condition for total stability, we note the following obvious rephrasing of Definition 2.1.

3.1. PROPOSITION.

3.1.1 $(\varphi_0, x_0) \in G(X)\times X$ is $\mathfrak{G}$-totally stable iff $\forall \mathscr{U} \in \mathscr{N}_X(x_0)$

$$\exists \mathscr{W} \in \mathscr{N}_{\mathscr{G}}(\varphi_0) : \forall \varphi \in \mathscr{W} \Rightarrow \exists \mathscr{V} \in \mathscr{N}_{\mathfrak{X}}(x_0) : \varphi\mathscr{V} \subset \mathscr{V}.$$

3.1.2 $(\varphi_0, x_0) \in G(X)\times X$ is strongly $\mathfrak{G}$-totally stable iff

$$\forall \mathscr{U} \in \mathscr{N}_X(x_0) \quad \exists \mathscr{W} \in \mathscr{N}_{\mathscr{G}}(\varphi_0) \quad , \quad \exists \mathscr{V} \in \mathscr{N}_{\mathfrak{X}}(x_0) : \mathscr{W}(\mathscr{V}) \subset \mathscr{V}.$$

We also note that, if S is a topological semigroup, a base of neighbourhoods of the identity generates, via translations, a base of neighbourhoods for any other element.

3.2. THEOREM. *Let* X *be a topological vector space, let* $\mathfrak{G}$ *be a topology on* $G(X)\times X$ *such that* $\mathscr{G}$ *is a topological semigroup and there is a base of neighbourhoods of the identity, each of which contains a translation by vectors in some neighbourhood of the origin; in these hypotheses if* (φ_0, x_0) *is strongly* $\mathfrak{G}$*-totally stable, there exists a base of* φ_0*-contracting neighbourhoods of* x_0.

Proof. Assuming strong total stability, in view of Proposition 3.1.2,

$$\exists \mathscr{W}^* \in \mathscr{N}_{\mathscr{G}}(id), \exists \mathscr{V} \in \mathscr{N}_{\mathfrak{X}}(x_0) : \mathscr{W}^* \circ \varphi_0 = : \mathscr{W} \in \mathscr{N}_{\mathscr{G}}(\varphi_0), \quad \mathscr{W}(\overline{\mathscr{V}}) \subset \overline{\mathscr{V}};$$

if $\mathscr{V}$ is not φ_0-contracting, this means that $\exists \bar{x} \in \overline{\mathscr{V}} : \varphi_0 \bar{x} \in \partial\mathscr{V}$. Choose now an arbitrary neighbourhood $\mathscr{B}$ of $\varphi_0 \bar{x}$ and $y \in \mathscr{B} \setminus \overline{\mathscr{V}}$ such that $k : z \mapsto z + (y - \varphi_0 \bar{x})$, $k \in \mathscr{W}^*$; but then $k \circ \varphi_0 \bar{x} = y \notin \overline{\mathscr{V}}$ contrary to the hypothesis.

3.3. *Remark.*

3.3.1. If X is a metric vector space and $\mathscr{T} = \mathscr{D}_0$, Theorem 3.2 is obviously applicable (yielding actually a base of strictly contracting neighbourhoods). In particular this holds also if $G(X) = \mathrm{Lip}(X)$ and $\mathscr{T} = \mathscr{L}_0$ By Theorem 2.5.1 then strong total stability with respect to topologies $\mathscr{D}_0$ $(\mathscr{L}_0)$, $\mathscr{K}_0(\mathscr{C}_0)$, $\mathscr{K}_1(\mathscr{C}_1)$ are equivalent concepts.

3.3.2. If X has a differentiable structure, the translation k is a smooth (as smooth as X is) diffeomorphism, so that $k \circ \varphi_0$ is as smooth as φ_0 is. Thus the theorem holds if we choose as $G(X)$ a semigroup of smooth maps. Note that in a local theory such as this X might be a manifold modelled on a topological vector space, whithout affecting the results.

4. Bifurcation Theory

In [3] there is some discussion on the relations between asymptotic behaviour of a flow, total stability and bifurcation theory, as well as on the interest of this theory in applications. We refer to that paper for motivations and more detailed insight in the background material.

4.1. Definition. Let X be a topological space and let $\mathscr{R}$ be a subset of 2^X (e.g. the set of compact subsets of X). Let $\mathscr{I}$ be a set of indices endowen with some topology (e.g. an interval of the real line) and assume a topology be given on $\mathscr{R}$. Let there be given a map $\Phi : \mathscr{I} \to G(X)$ $(\mu \mapsto \varphi_\mu)$ such that, if id is the identity map on X, the product map $\Phi \times id : \mathscr{I} \times X \to G(X) \times X$ is continuous in the topologies given on $\mathscr{I}$, X, $G(X) \times X$; assume a map $\mathscr{M} : \mathscr{I} \times \mathscr{R} \to \mathscr{R}$ be defined on a neighbourhood of $\mu_0 \in \mathscr{I}$, continuous in μ_0 and such that $\mu \to \mu_0 \Rightarrow \mathscr{M}(\mu) \to \mathscr{M}(\mu_0)$, $\varphi_\mu(\mathscr{M}_\mu) = \mathscr{M}_\mu$. We shall say that μ_0 is a bifurcation point if a second map $\mathscr{M}' : \mathscr{I} \to \mathscr{R}$ is defined on a neighbourhood of μ_0, such that $\mu \to \mu_0 \Rightarrow \mathscr{M}'(\mu) \to \mathscr{M}(\mu_0)$, $\varphi_\mu(\mathscr{M}'_\mu) = \mathscr{M}'_\mu$, $\mathscr{M}(\mu) \cap \mathscr{M}'(\mu) = \varnothing$ $\mu \neq \mu_0$.

4.2. *Remark.* If the evaluation map $ev : G(X) \times X \to X$ is continuous, then so is the map $\mathscr{I} \times X \to X : (\mu, x) \mapsto \varphi_\mu x$, being a composition of continuous maps.

4.3. Definition. We take as the topology on $\mathscr{R}$ the natural one, induced by X, i.e. a neighbourhood of $N \in \mathscr{R}$ is defined as

$$\mathfrak{U}_{\mathscr{U}} := \{N' \in \mathscr{R}, N' \subset \mathscr{U} ; N \subset \mathscr{U} \text{ open in } X\}$$

4.4. *Remark.* This topology will in general fail to be separated: e.g. take as $\mathscr{R}$ the compact subsets of X, then all neighbourhoods of $N \in \mathscr{R}$ are neighbourhoods of any compact subset of N (but not conversely, so that the topology, though not T_1, is T_0). This however is the natural notion of "nearness" and " convergence " of sets, as far as bifurcation theory goes. In [3] it was additionally assumed that the map $\mathcal{M}$ be Hausdorff continuous.

In [3] it has been proved that if $\mathcal{R}$ is the collection of compact subsets of a locally compact metric space X, if $\mathcal{I}$ is a (possibly one-sided) neighbourhood of $\mu_0 \in \mathbf{R}$ and $\mathcal{M}(\mu_0)$ is asymptotically stable, while $\mathcal{M}(\mu)$ is completely unstable (i.e. negatively asymptotically stable), then μ_0 is a bifurcation point. This generalized some classical results on the Hopf bifurcation [9]. Here we give a generalization of this result, in the case of discrete dynamical systems.

We first of all note that all results in the preceding sections easily extend to compact or closed invariant sets, instead of fixd points. Let $\mathcal{R}$ contain the closed sets or, if X is locally compact, the compact sets of X. We assume to have chosen a topology on $G(X) \times X$ such as those introduced in Sec. 2.

Let all notations correspond to those already introduced.

4.5. THEOREM.

4.5.1. *Let $\mathcal{M} : \mathcal{I} \to \mathcal{R}$ be continuous at μ_0 and let $\mathcal{M}(\mu_0)$ have a base of φ_{μ_0}-contracting neighbourhoods. Let $\mathcal{E}$ be a neighbourhood of μ_0 such that $\mathcal{M}(\mu)$ is completely unstable for any $\mu \in \mathcal{E} \setminus \{\mu_0\}$, i.e.*

$$\forall \mu \in \mathcal{E} \setminus \{\mu_0\} \; \exists S_\mu \in \mathcal{N}(\mathcal{M}(\mu)) : y \in S_\mu \setminus \mathcal{M}(\mu) \Rightarrow \exists n_\mu(y) : n \geq n_\mu(y) \Rightarrow \varphi_\mu^n y \notin S_\mu .$$

Assume moreover that for any x in a suitable neighbourhood of $\mathcal{M}(\mu_0)$ and all $\mu \in \mathcal{E}$

$$\omega(x) := \bigcap_{N=0}^{\infty} \overline{\bigcup_{n=N}^{\infty} \varphi_\mu^n x} \neq \varnothing$$

(assume that all φ_μ-orbits of such x are precompact). Then μ_0 is a bifurcation point and the sets $\mathcal{M}'_\mu$ are weakly attracting (the sets $\mathcal{M}'_\mu$, are attracting).

4.5.2. *If moreover X is locally compact, the sets $\mathcal{M}'_\mu$ may be chosen to be asymptotically stable.*

Proof. Fix any (sufficiently small) neighbourhood of $\mathcal{M}(\mu_0)$ and let $\mathcal{V}$ be a corresponding φ_{μ_0}-contracting neighbourhood. In view of the proof of Theorem 2.5, if μ is near enough to μ_0, $\mathcal{V}$ will also be φ_μ-contracting (actually in the $\mathcal{K}_2(\mathcal{M}(\mu_0))$ or $\mathcal{C}_2(\mathcal{M}(\mu_0))$ case $\mathcal{V}$ will contain another neighbourhood $\mathcal{V}'$ which will be φ_μ-contracting, which is enough for our purposes). Let $\mathcal{V}^*$ be a corresponding φ_μ-asymptotically stable neighbourhood (cfr. Lemma 2.8). If μ is close enough to μ_0 we shall have $\mathcal{M}(\mu) \subset \mathcal{V}^*$ and we can always assume $S_\mu \subset \mathcal{V}^*$ (possibly choosing a smaller neighbourhood as S_μ). It follows that $\overline{\mathcal{V}^* \setminus S_\mu}$ is positively invariant and asymptotically stable. If this set is actually invariant call it $\mathcal{M}'(\mu)$ and the proof is through.

Otherwise set $\mathcal{M}'(\mu) := \max \{\mathcal{H} \subset \overline{\mathcal{V}^* \setminus S_\mu} : \varphi_\mu \mathcal{H} = \mathcal{H}\}$; $\mathcal{M}'(\mu) \neq \varnothing$ since $\omega(S_\mu) \neq \varnothing$ by hypothesis, while for any x in the region of attraction of $\overline{\mathcal{V}^* \setminus S_\mu}$, $\omega(x) \subset \mathcal{M}'(\mu)$, which proves the weak attractivity; if the orbit of x is precompact $\varphi_\mu^n x \to \omega(x)$ as $n \to \infty$ and the Proof of 4.5.1 is complete.

If X is locally compact, from standard theorems (e.g. Theorem V, 5.12 in [8]) the same construction yields that $\mathcal{M}'(\mu)$ is asymptotically stable with respect to $\overline{\mathcal{V}^* \setminus S_\mu}$ (which can now be assumed to be compact) and hence, by the same argument as in [3], asymptotically stable.

4.6. *Remark*. The request of some compactness property for orbits is obviously necessary if one wants some attractiveness property for the bifurcated sets. If X is locally compact these properties are trivially satisfied, otherwise they might be hard to realize, but in a number of applications this can be obtained (see [10]).

4.7. *Example*. Through a small modification of a classical example it is easy to exhibit the behaviour described in Theorem 4.5.

Let $X = \mathbf{R}^2$, $\mathcal{I} = [0, \alpha)$ and let φ_μ be the time-1 map induced by the following differential system

$$\dot{x} = -x^3 \sin \frac{\pi}{x^2+y^2} - y + \mu x$$

$$\dot{y} = x + \mu y .$$

Consider the (Lyapunov) function $\Psi(x, y) := r^2/2 := \dfrac{x^2+y^2}{2}$

$$\dot{\Psi} = -x^4 \sin \frac{\pi}{x^2+y^2} + \mu (x^2 + y^2) \geq -r^4 + \mu r^2 > 0 \quad \text{for } r^2 < \mu .$$

When $\mu = 0$ there is a base of contracting neighbourhoods, while for $\mu > 0$ the origin is completely unstable, so that a bifurcated attracting orbit must appear outside a ball of radius $\mu^{1/2}$.

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