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Oscillatory and asymptotic properties of differential equations with deviating arguments

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Equazioni differenziali ordinarie. — *Oscillatory and asymptotic properties of differential equations with deviating arguments* (*). Nota di I. P. STAVROULAKIS, presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — Si studia il comportamento asintotico e in particolare l'oscillatorietà delle soluzioni dell'equazione differenziale con argomento ritardato (D). Si dànno condizioni affinché tutte le soluzioni siano oscillatorie o infinitesime per $t \to \infty$. Si classificano le soluzioni di (D) in base al loro comportamento per $t \to \infty$ e al loro carattere oscillatorio. I risultati ottenuti estendono quelli recenti di Marusiak [3] e di Staikos e Sficas [4].

This paper is concerned with the oscillatory and asymptotic behavior of solutions of the differential equation with deviating arguments

$$\text{(D)} \qquad [r(t)\, x^{(n-N)}(t)]^{(N)} + f(t\,; x[\tau_1(t)], x[\tau_2(t)], \cdots, x[\tau_m(t)]) = 0, \quad t \geqq t_0$$

where N, n, m are positive integers with $2 \leqq N \leqq n - 1$.

We suppose that $\tau_j\ (j = 1, 2, \cdots, m)$, f and r are continuous real-valued functions and such that:

(i) $\tau_j\ (j = 1, 2, \cdots, m)$ are defined on the half-line $[t_0, \infty)$ with

$$\lim_{t\to\infty} \tau_j(t) = \infty.$$

(ii) f is defined on $[t_0, \infty) \times \mathbf{R}^m$.

$$f(t\,; 0, 0, \cdots, 0) = 0 \qquad \text{for every} \quad t \geqq t_0$$

and

$$\int^{\infty} t^{N-1} f(t\,; c, c, \cdots, c)\, dt = \pm \infty \qquad \text{for every non-zero constant } c.$$

(iii) r is nonnegative on $[t_0, \infty)$ and such that

$$\int^{\infty} \frac{dt}{r(t)} = \infty.$$

Throughout this paper, by "solution" of the differential equation (D) we shall mean only solutions x which are defined on the half-line $[t_x, \infty)$. The oscillatory character is considered in the usual sense, i.e. a solution is called *oscillatory* if it has no last zero, otherwise it is called *nonoscillatory*.

To obtain our results we need the following three lemmas. Lemmas 1 and 3 are given in [5], while Lemma 2 is a unified adaptation of two lemmas due to Kiguradze (cfr. [2] and [1]).

(*) This paper is a part of the Author's Doctoral Thesis submitted to the School of Physics and Mathematics of the University of Ioannina.

(**) Nella seduta dell'8 maggio 1976.

LEMMA 1. *Consider the linear differential equation*

$$z' - \frac{\mu}{t} z + \frac{h(t)}{t} = 0 \tag{L}$$

where μ *is a positive integer and* h *is continuous on* $[T, \infty)$, $T > 0$.

If $\lim_{t\to\infty} |h(t)| = \infty$ *and* u *is the solution of* (L) *with* $u(T) = 0$, *then*

$$\lim_{t\to\infty} |u(t)| = \infty.$$

LEMMA 2. *Let* u *be a positive* n*-times differentiable function on an interval* $[a, \infty)$. *If* $u^{(n)}$ *is of constant sign and not identically zero for all large* t, *then there exist a* $t_u \geqq a$ *and an integer* l, $0 \leqq l \leqq n$ *with* $n + l$ *odd for* $u^{(n)} \leqq 0$ *or* $n + l$ *even for* $u^{(n)} \geqq 0$ *and such that for every* $t \geqq t_u$

$$l > 0 \Rightarrow u^{(k)}(t) > 0 \qquad (k = 0, 1, \cdots, l - 1)$$

and

$$l \leqq n - 1 \Rightarrow (-1)^{l+k} u^{(k)}(t) > 0 \qquad (k = l, l + 1, \cdots, n - 1).$$

LEMMA 3. *If* u *is as in Lemma 2 and for some* $k = 0, 1, \cdots, n - 2$

$$\lim_{t\to\infty} u^{(k)}(t) = c, \qquad c \in \mathbf{R}$$

then

$$\lim_{t\to\infty} u^{(k+1)}(t) = 0.$$

THEOREM 1. *Let the functions* f, τ_j *and* r *satisfy the conditions* (i)-(iii) *and in addition suppose that*:

(C_1) $f(t; y_1, y_2, \cdots, y_m)$ *is nondecreasing in each variable* $y_1, y_2, \cdots, y_m$.

(C_2) *For every constant* $c \neq 0$ *and any integer* k, $0 \leqq k \leqq N - 2$

$$\int^{\infty} t^{N-2-k} f(t; c\rho(t)[\tau_1(t)]^{n-N+k}, \cdots, c\rho(t)[\tau_m(t)]^{n-N+k})\, dt = \pm\infty$$

where

$$\rho(t) = \frac{1}{\max\limits_{t/2 \leqq s \leqq t} r(s)}.$$

Then for n *even all solutions of* (D) *are oscillatory, while for* n *odd every solution of* (D) *is either oscillatory or tending monotonically to zero as* $t \to \infty$ *together with ist first* $n - 1$ *derivatives.*

Proof. Let x be a nonoscillatory solution of (D) with $\lim_{t\to\infty} x(t) \neq 0$. Since the substitution $u = -x$ transforms (D) into an equation of the same form satisfying the assumptions of the theorem we can suppose, without loss of generality, that $x(t) > 0$ for every $t \geqq t_0$.

By (i), we can choose a $t_1 \geqq t_0$ such that for every $t \geqq t_1$

$$\tau_j(t) \geqq t_0 \qquad (j = 1, 2, \cdots, m).$$

Obviously, $\mu = \min\{\inf_{t \geqq t_1} x[\tau_1(t)], \cdots, \inf_{t \geqq t_1} x[\tau_m(t)]\} > 0$ and, in view of (C_1) and (ii), for every $t \geqq t_1$ we have

$$f(t; x[\tau_1(t)], x[\tau_2(t)], \cdots, x[\tau_m(t)]) \geqq f(t; \mu, \mu, \cdots, \mu) \geqq 0. \tag{1}$$

If

$$q_{ij}(t) = \int_{t_1}^{t} s^i [r(s) x^{(n-N)}(s)]^{(j)} \, ds, \qquad 1 \leqq i \leqq j$$

then, integrating by parts, we obtain

$$q_{ij}(t) = t q'_{i-1,j-1}(t) - i q_{i-1,j-1}(t) - t_1^i [r(s) x^{(n-N)}(s)]_{s=t_1}^{(j-1)}.$$

Hence $q_{i-1,j-1}$ is a solution of the linear differential equation

$$z' - \frac{i}{t} z + \frac{h_{ij}(t)}{t} = 0 \tag{L_{ij}}$$

where $h_{ij}(t) = -q_{ij}(t) - t_1^i [r(s) x^{(n-N)}(s)]_{s=t_1}^{(j-1)}$. It is clear that this solution satisfies the initial condition $q_{i-1,j-1}(t_1) = 0$.

For $i = N - 1$ and $j = N$ we get

$$h_{N-1,N}(t) = -q_{N-1,N}(t) - t_1^{N-1} [r(s) x^{(n-N)}(s)]_{s=t_1}^{(N-1)} =$$

$$= \int_{t_1}^{t} - s^{N-1} [r(s) x^{(n-N)}(s)]^{(N)} \, ds - t_1^{N-1} [r(s) x^{(n-N)}(s)]_{s=t_1}^{(N-1)} =$$

$$= \int_{t_1}^{t} s^{N-1} f(s; x[\tau_1(s)], x[\tau_2(s)], \cdots, x[\tau_m(s)]) \, ds -$$

$$- t_1^{N-1} [r(s) x^{(n-N)}(s)]_{s=t_1}^{(N-1)}$$

and consequently, by (1), we obtain

$$h_{N-1,N}(t) \geqq \int_{t_1}^{t} s^{N-1} f(s; \mu, \mu, \cdots, \mu) \, ds - t_1^{N-1} [r(s) x^{(n-N)}(s)]_{s=t_1}^{(N-1)}$$

which, in view of (ii), yields

$$\lim_{t\to\infty} | h_{N-1,N}(t) | = \infty .$$

Now, applying Lemma 1 to the differential equation $(L_{N-1,N})$, we obtain

$$\lim_{t\to\infty} | q_{N-2,N-1}(t) | = \infty .$$

Since

$$h_{N-2,N-1}(t) = -q_{N-2,N-1}(t) - t_1^{N-2} [r(s)\, x^{(n-N)}(s)]_{s=t_1}^{(N-2)}$$

we have

$$\lim_{t\to\infty} | h_{N-2,N-1}(t) | = \infty$$

and, applying the same lemma to the differential equation $(L_{N-2,N-1})$, we get

$$\lim_{t\to\infty} | q_{N-3,N-2}(t) | = \infty .$$

Following the same procedure, we finally obtain

$$\lim_{t\to\infty} | q_{01}(t) | = \infty$$

which, by virtue of

$$q_{01}(t) = \int_{t_1}^{t} s^0 [r(s)\, x^{(n-N)}(s)]' \, ds = r(t)\, x^{(n-N)}(t) - r(t_1)\, x^{(n-N)}(t_1)$$

yields

$$\lim_{t\to\infty} r(t)\, x^{(n-N)}(t) = \pm \infty .$$

We shall show that $\lim_{t\to\infty} r(t)\, x^{(n-N)}(t) = \infty$. Indeed, if $\lim_{t\to\infty} r(t)\, x^{(n-N)}(t) = -\infty$ then there exists a $t_2 \geqq t_1$ such that for every $t \geqq t_2$

$$r(t)\, x^{(n-N)}(t) < -1 \qquad \text{or} \qquad x^{(n-N)}(t) < -\frac{1}{r(t)} .$$

Thus

$$x^{(n-N-1)}(t) < x^{(n-N-1)}(t_2) - \int_{t_2}^{t} \frac{ds}{r(s)}$$

which, by (iii), gives $\lim_{t\to\infty} x^{(n-N-1)}(t) = -\infty$. But this implies that $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the positivity of x.

By $\lim_{t\to\infty} r(t)\, x^{(n-N)}(t) = \infty$ we have

$$r(t)\, x^{(n-N)}(t) > 1 \qquad \text{or} \qquad x^{(n-N)}(t) > \frac{1}{r(t)}$$

for some $t_3 \geqq t_1$ and every $t \geqq t_3$. Hence

$$x^{(n-\mathrm{N}-1)}(t) > x^{(n-\mathrm{N}-1)}(t_3) + \int_{t_3}^{t} \frac{ds}{r(s)}$$

which, in view of (iii), yields

$$\lim_{t\to\infty} x^{(n-\mathrm{N}-1)}(t) = \infty.$$

From this, it is clear that

$$\lim_{t\to\infty} x^{(j)}(t) = \infty \qquad (j = 0, 1, \cdots, n - \mathrm{N} - 1)$$

and consequently for all large t

$$x^{(j)}(t) > 0 \qquad (j = 0, 1, \cdots, n - \mathrm{N} - 1).$$

Moreover, for every $t \geqq t_1$ we have

$$y^{(\mathrm{N})}(t) = -f(t; x[\tau_1(t)], x[\tau_2(t)], \cdots, x[\tau_m(t)]) \leqq$$
$$\leqq -f(t; \mu, \mu, \cdots, \mu) \leqq 0$$

where $y \equiv r x^{(n-\mathrm{N})}$. We notice that, since $f(t; \mu, \mu, \cdots, \mu)$ is, by (ii), not identically zero for all large t the same holds for $y^{(\mathrm{N})}(t)$. On the other hand $\lim_{t\to\infty} y(t) = \infty$ and therefore $y(t) > 0$ for all large t. Thus, applying Lemma 2 to the function y, we derive that there exists an integer l, $0 \leqq l \leqq \mathrm{N}$ with $\mathrm{N} + l$ odd and such that for some t_y and every $t \geqq t_y$

$$l > 0 \Rightarrow y^{(i)}(t) > 0 \qquad (i = 0, 1, \cdots, l - 1)$$

and

$$l \leqq \mathrm{N} - 1 \Rightarrow (-1)^{l+i} y^{(i)}(t) > 0 \quad (i = l, l+1, \cdots, \mathrm{N} - 1)$$

Since the integer $\mathrm{N} + l$ is odd, obviously $l \leqq \mathrm{N} - 1$. Furthermore, by $\lim_{t\to\infty} y(t) = \infty$, we conclude that $l > 0$, thus for every $t \geqq t_y$ we have

$$\left\{\begin{array}{ll} y^{(i)}(t) > 0 & (i = 0, 1, \cdots, l - 1) \\ (-1)^{l+i} y^{(i)}(t) > 0 & (i = l, l+1, \cdots, \mathrm{N} - 1). \end{array}\right. \tag{3}$$

Consequently, by Taylor's formula, for $t \geqq 2\,t_y$ we get

$$x(t) = x\left(\frac{t}{2}\right) + \frac{\left(t - \frac{t}{2}\right) x'\left(\frac{t}{2}\right)}{1!} + \cdots + \frac{\left(t - \frac{t}{2}\right)^{n-\mathrm{N}} x^{(n-\mathrm{N})}(t^*)}{(n - \mathrm{N})!} =$$

$$= x\left(\frac{t}{2}\right) + \frac{\left(t - \frac{t}{2}\right) x'\left(\frac{t}{2}\right)}{1!} + \cdots + \frac{\left(t - \frac{t}{2}\right)^{n-\mathrm{N}} y(t^*)}{(n - \mathrm{N})!\, r(t^*)}, \quad \frac{t}{2} \leqq t^* \leqq t$$

and since the function y is nondecreasing for every $t \geqq 2\,t_y$ we have

$$x(t) \geqq x\left(\frac{t}{2}\right) + \frac{\left(t - \frac{t}{2}\right) x'\left(\frac{t}{2}\right)}{1!} + \cdots + \frac{\left(t - \frac{t}{2}\right)^{n-N-1} x^{(n-N-1)}\left(\frac{t}{2}\right)}{(n-N-1)!} + $$

$$+ \rho(t) \frac{\left(t - \frac{t}{2}\right)^{n-N}}{(n-N)!} y\left(\frac{t}{2}\right).$$

From the last inequality, in view of (2), for every $t \geqq 2\,t_y$ we obtain

$$x(t) \geqq c_1\, \rho(t)\, t^{n-N} y\left(\frac{t}{2}\right) \tag{4}$$

where the constant c_1 is positive.

Again, by Taylor's formula, for every $t \geqq 4\,t_y$ we have

$$y\left(\frac{t}{2}\right) = y(2\,t_y) + \frac{\left(\frac{t}{2} - 2\,t_y\right) y'(2\,t_y)}{1!} + \cdots + \frac{\left(\frac{t}{2} - 2\,t_y\right)^{l-1} y^{(l-1)}(t_*)}{(l-1)!}, \qquad 2\,t_y \leqq t_* \leqq \frac{t}{2}$$

and since $y^{(l-1)}$ is nondecreasing

$$y\left(\frac{t}{2}\right) \geqq y(2\,t_y) + \frac{\left(\frac{t}{2} - 2\,t_y\right) y'(2\,t_y)}{1!} + \cdots + \frac{\left(\frac{t}{2} - 2\,t_y\right)^{l-1} y^{(l-1)}(2\,t_y)}{(l-1)!}.$$

Thus, by (3), there exist a positive constant c_2 and a $t_4 \geqq 4\,t_y$ such that for every $t \geqq t_4$

$$y\left(\frac{t}{2}\right) \geqq c_2\, t^{l-1}. \tag{5}$$

By (4) and (5), we get

$$x(t) \geqq c\rho(t)\, t^{n-N+k}, \qquad t \geqq t_4 \tag{6}$$

where $c = c_1 \cdot c_2$, $k = l - 1$ and $0 \leqq k \leqq N - 2$.

We choose a $t_5 \geqq t_4$ such that for every $t \geqq t_5$

$$\tau_j(t) \geqq t_4 \qquad (j = 1, 2, \cdots, m).$$

Thus, in view of (C_1), (ii) and (6), for every $t \geqq t_5$ we have

$$f(t; x[\tau_1(t)], \cdots, x[\tau_m(t)]) \geqq \tag{7}$$

$$\geqq f(t; c\rho(t)[\tau_1(t)]^{n-N+k}, \cdots, c\rho(t)[\tau_m(t)]^{n-N+k}) \geqq 0.$$

Now, we consider again the functions

$$q_{ij}(t) = \int_{t_5}^{t} s^i \, [r(s)\, x^{(n-\mathrm{N})}(s)]^{(j)} \, ds, \qquad 1 \leqq i \leqq j$$

and following the same procedure as before, by virtue of (7) and (C_2), we obtain

$$\lim_{t\to\infty} | q_{0,k+2}(t) | = \infty .$$

This relation, in view of

$$q_{0,k+2}(t) = \int_{t_5}^{t} s^0 \, [r(s)\, x^{(n-\mathrm{N})}(s)]^{(k+2)} \, ds$$

$$= [r(t)\, x^{(n-\mathrm{N})}(t)]^{(k+1)} - [r(s)\, x^{(n-\mathrm{N})}(s)]^{(k+1)}_{s=t_5}$$

leads to

$$\lim_{t\to\infty} [r(t)\, x^{(n-\mathrm{N})}(t)]^{(k+1)} = \pm\infty \quad \text{i.e.} \quad \lim_{t\to\infty} y^{(l)}(t) = \pm\infty$$

which contradicts (3). Thus every solution of (D) is oscillatory or such that $\lim_{t\to\infty} x(t) = 0$ monotonically. In the last case, by Lemma 3, we get

$$\lim_{t\to\infty} x^{(k)}(t) = 0 \qquad (k = 0, 1, \cdots, n-1). \tag{8}$$

To complete the proof we have to show that (8) can occur only when n is odd. To do this let x be a solution of (D) which satisfies (8). We again assume, without loss of generality, that x is positive. For $y \equiv r x^{(n-\mathrm{N})}$, (D) in view of ($C_1$) and (ii), yields $y^{(\mathrm{N})}(t) \leqq 0$ for all large t. Hence, the function y is monotone and therefore $\lim_{t\to\infty} y(t) \in \mathbf{R}^*$. If we now suppose that $\lim_{t\to\infty} y(t) \neq 0$ then for some positive constant M, we have

$$y(t) > \mathrm{M} \qquad \text{for all large } t$$

or

$$y(t) < -\mathrm{M} \qquad \text{for all large } t.$$

If the first of the above cases is valid, that is

$$x^{(n-\mathrm{N})}(t) > \frac{\mathrm{M}}{r(t)} \qquad \text{for all large } t$$

then, by integration, in view of (iii), we obtain

$$\lim_{t\to\infty} x^{(n-\mathrm{N}-1)}(t) = \infty$$

which leads to the contradiction

$$\lim_{t\to\infty} x(t) = \infty$$

since $\lim_{t\to\infty} x(t) = 0$.

If the second case is valid, that is

$$x^{(n-\mathrm{N})}(t) < -\frac{\mathrm{M}}{r(t)}$$

then, analogously, it leads to the contradiction $\lim_{t\to\infty} x(t) = -\infty$. Hence, it has been proved that $\lim_{t\to\infty} y(t) = 0$. In addition we remark that the case

$$y(t) = 0 \quad \text{i.e.} \quad x^{(n-\mathrm{N})}(t) = 0 \qquad \text{for all large } t$$

is impossible, since in this case the solution $x(t)$ coincides with a polynomial of t and, as bounded, it would be constant which contradicts (8). We now consider the following two possible cases:

1) $y(t) > 0$ for all large t. In this case we have

$$y^{(\mathrm{N})}(t) \leqq 0 \quad \text{and} \quad y'(t) < 0 \qquad \text{for all large } t$$

and consequently, by Lemma 2, we conclude that the integer N is odd. Moreover

$$x(t) > 0 \quad , \quad x'(t) < 0 \quad \text{and} \quad x^{(n-\mathrm{N})}(t) > 0 \quad \text{for all large } t$$

thus, by Lemma 2, we obtain that the integer $n - \mathrm{N}$ is even. Hence the integer $n = (n - \mathrm{N}) + \mathrm{N}$ is odd.

2) $y(t) < 0$ for all large t. We have

$$y^{(\mathrm{N})}(t) \leqq 0 \quad \text{and} \quad y'(t) > 0 \qquad \text{for all large } t$$

thus, applying Lemma 2 to the function $u = -y$, we conclude that the integer N is even. Furthermore

$$x(t) > 0 \quad , \quad x'(t) < 0 \quad \text{and} \quad x^{(n-\mathrm{N})}(t) < 0 \quad \text{for all large } t$$

and therefore, by Lemma 2, the integer $n - \mathrm{N}$ is odd. Hence the integer $n = (n - \mathrm{N}) + \mathrm{N}$ is again odd. This completes the proof of the theorem.

Remark 1. The above theorem extends a result due to Marušiak [3, Theorem 3.1] as well as a result due to Staikos and Sficas [4, Theorem 3], concerning the oscillatory properties of the differential equation

$$x^{(n)}(t) + f(t, x(t), x[\tau(t)]) = 0. \tag{9}$$

THEOREM 2. *Let the differential equation* (D) *with* $N \geqq 3$ *be subject to the conditions* (i)-(iii) *and in addition suppose that*:

(C_1') $f(t; y_1, y_2, \cdots, y_n)$ *is nonincreasing in each variable* $y_1, y_2, \cdots, y_m$.

(C_2') *For every constant* $c \neq 0$ *and any integer* $k, 0 \leqq k \leqq N-3$

$$\int^{\infty} t^{N-2-k} f(t; c\rho(t)[\tau_1(t)]^{n-N+k}, \cdots, c\rho(t)[\tau_m(t)]^{n-N+k})\,dt = \mp\infty$$

and

$$\int^{\infty} f(t; c\rho(t)[\tau_1(t)]^{n-1}, \cdots, c\rho(t)[\tau_m(t)]^{n-1})\,dt = \mp\infty$$

where

$$\rho(t) = \frac{1}{\max\limits_{t/2 \leqq s \leqq t} r(s)}.$$

Then every solution x *of* (D) *satisfies exactly one of the following*:

(I) x *is oscillatory*.

(II) x *and its first* $n-1$ *derivatives tend monotonically to zero as* $t \to \infty$.

(III) *It holds*

$$\lim_{t\to\infty} [r(t)\, x^{(n-N)}(t)]^{(i)} = \infty \qquad (i = 0, 1, \cdots, N-1) \textit{ and}$$

$$\lim_{t\to\infty} x^{(j)}(t) = \infty \qquad (j = 0, 1, \cdots, n-N-1)$$

or

$$\lim_{t\to\infty} [r(t)\, x^{(n-N)}(t)]^{(i)} = -\infty \qquad (i = 0, 1, \cdots, N-1) \textit{ and}$$

$$\lim_{t\to\infty} x^{(j)}(t) = -\infty \qquad (j = 0, 1, \cdots, n-N-1).$$

Moreover (II) *occurs only in the case of even* n.

Proof. Let x be a nonoscillatory solution of (D) with $\lim\limits_{t\to\infty} x(t) \neq 0$. As in the proof of Theorem 1, we suppose, without loss of generality, that $x(t) > 0$ for every $t \geqq t_0$. Moreover, by (i), (ii) and (C_1'), we can choose a $t_1 \geqq t_0$ such that for every $t \geqq t_1$

$$f(t; x[\tau_1(t)], x[\tau_2(t)], \cdots, x[\tau_m(t)]) \leqq f(t; \mu, \mu, \cdots, \mu) \leqq 0 \tag{10}$$

where μ is defined as in (1).

If we consider the functions

$$q_{ij}(t) = \int_{t_1}^{t} s^i \left[r(s)\, x^{(n-N)}(s)\right]^{(j)} ds, \qquad 1 \leqq i \leqq j$$

then following the same procedure as in the preceding theorem, and taking into account (10), (ii) and (iii), we obtain

$$\lim_{t\to\infty} r(t)\, x^{(n-N)}(t) = \infty \qquad \text{and} \qquad \lim_{t\to\infty} x^{(n-N-1)}(t) = \infty$$

from which we can easily derive (2).

Moreover, for every $t \geqq t_1$ we have

$$y^{(N)}(t) = -f(t; x[\tau_1(t)], x[\tau_2(t)], \cdots, x[\tau_m(t)]) \geqq -f(t; \mu, \mu, \cdots, \mu) \geqq 0$$

where $y = rx^{(n-N)}$. Since $f(t; \mu, \mu, \cdots, \mu)$ is, by (ii), not identically zero for all large t the same holds for $y^{(N)}(t)$. By $\lim_{t\to\infty} y(t) = \infty$ we have $y(t) > 0$ for all large t, and hence applying Lemma 2 to the function y, we conclude that there exists an integer l, $0 \leqq l \leqq N$ with $N + l$ even and such that for some t_y and every $t \geqq t_y$

$$l > 0 \Rightarrow y^{(i)}(t) > 0 \qquad (i = 0, 1, \cdots, l-1)$$

and

$$l \leqq N - 1 \Rightarrow (-1)^{l+i} y^{(i)}(t) > 0 \quad (i = l, l+1, \cdots, N-1).$$

Furthermore, by $\lim_{t\to\infty} y(t) = \infty$, we conclude that $l > 0$. Now we consider the following two cases:

Case 1. $l = N$. In this case we have

$$y^{(i)}(t) > 0 \qquad \text{for every} \quad t \geqq t_y \qquad (i = 0, 1, \cdots, N-1).$$

Thus, as in the proof of Theorem 1, we obtain (6), which can be written

$$x(t) \geqq c\rho(t)\, t^{n-1} \qquad \text{for every} \quad t \geqq t_4. \tag{11}$$

If now $t_5 \geqq t_y$ is such that

$$\tau_j(t) \geqq t_4 \qquad (j = 1, 2, \cdots, m)$$

then, by (C_1'), (ii) and (11), for every $t \geqq t_5$ we get

$$f(t; x[\tau_1(t)], \cdots, x([\tau_m(t)]) \leqq$$
$$\leqq f(t; c\rho(t)[\tau_1(t)]^{n-1}, \cdots, c\rho(t)[\tau_m(t)]^{n-1}) \leqq 0$$

and consequently, by (D)

$$[r(t)\, x^{(n-N)}(t)]^{(N)} + f(t; c\rho(t)[\tau_1(t)]^{n-1}, \cdots, c\rho(t)[\tau_m(t)]^{n-1}) \geqq 0.$$

Integrating from t_5 to t we obtain

$$[r(t)\,x^{(n-N)}(t)]^{(N-1)} \geqq -\int_{t_5}^{t} f(s\,;c\rho(s)\,[\tau_1(s)]^{n-1},\cdots,c\rho(s)\,[\tau_m(s)]^{n-1})\,ds + \\ + [r(s)\,x^{(n-N)}(s)]^{(N-1)}_{s=t_5}$$

which, by (C_2'), leads to

$$\lim_{t\to\infty} [r(t)\,x^{(n-N)}(t)]^{(N-1)} = \infty$$

and therefore

(12) $$\lim_{t\to\infty} [r(t)\,x^{(n-N)}(t)]^{(i)} = \infty \qquad (i = 0, 1, \cdots, N-1).$$

Furthermore, by $\lim_{t\to\infty} r(t)\,x^{(n-N)}(t) = \infty$ and (iii), we obtain

$$\lim_{t\to\infty} x^{(n-N-1)}(t) = \infty$$

hence

$$\lim_{t\to\infty} x^{(j)}(t) = \infty \qquad (j = 0, 1, \cdots, n-N-1).$$

The last relation and (12) imply that the solution x satisfies (III).

Case 2. $0 < l \leqq N-1$. We have

$$y^{(N-1)}(t) < 0 \qquad \text{for every} \quad t \geqq t_y.$$

Following the same procedure as in the proof of Theorem 1, setting $y^{(N-1)}$ in place of $y^{(N)}$, we obtain relation (6), i.e.

(13) $$x(t) \geqq c\rho(t)\,t^{\,n-N+k} \qquad \text{for every} \quad t \geqq t_4$$

where $0 \leqq k \leqq N-3$. Consequently, by (C_1'), (ii) and (13), we get

(14) $$f(t\,;x[\tau_1(t)],\cdots,x[\tau_m(t)]) \leqq \\ \leqq f(t\,;c\rho(t)\,[\tau_1(t)]^{n-N+k},\cdots,c\rho(t)\,[\tau_m(t)]^{n-N+k}) \leqq 0$$

for every $t \geqq t_5$, where t_5 has been chosen as in (7).

Next, by using the functions

$$q_{ij}(t) = \int_{t_5}^{t} s^i\,[r(s)\,x^{(n-N)}(s)]^{(j)}\,ds, \qquad 1 \leqq i \leqq j$$

taking into account (C_2') and (14) and working as in the proof of the previous theorem, we obtain

$$\lim_{t\to\infty} y^{(l)}(t) = \pm\infty$$

which leads to a contradiction. Hence, every nonoscillatory solution x of (D) satisfies (III) or $\lim_{t\to\infty} x(t) = 0$. In the last case, by Lemma 3, we have

$$\lim_{t\to\infty} x^{(k)}(t) = 0 \qquad (k = 0, 1, \cdots, n-1)$$

that is, x satisfies (II). Hence, every solution x of (D) satisfies exactly one of the properties (I), (II), (III). Finally, as in the last part of the proof of Theorem 1, we conclude that (II) occurs only in the case of even n.

Remark 2. The above Theorem 2 extends a recent result due to Staikos and Sficas [4, Theorem 4] concerning the differential equation (9).

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