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Asymptotic behaviour of perturbed nonlinear systems

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Equazioni differenziali ordinarie. — *Asymptotic behaviour of perturbed nonlinear systems.* Nota di M. BASTI e B. S. LALLI (*), presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — Gli Autori stabiliscono alcune relazioni asintotiche tra le soluzioni dei sistemi non perturbati

(*) $$x' = f(t, x), \quad (' = dx/dt)$$

e quelle dei sistemi perturbati

(**) $$y' = f(t, y) + g(t, y, Ty)$$

estendendo il concetto di equivalenza asintotica generalizzata di P. Talpalaru. Successivamente stabiliscono un teorema di equivalenza asintotica generalizzata tra i sistemi (*), (**).

1. INTRODUCTION

Recently, Brauer [5], Brauer and Wong [9], Fennel and Proctor [4], Hallam [6], Pachpatte [2], Talpalaru [3], and Marlin and Strubbe [1] among others have obtained general results on asymptotic behaviour of solutions of perturbed nonlinear systems. The purpose of this paper is to investigate these problems further. We are mainly interested in establishing asymptotic relationship between the solutions of unperturbed systems

$$x' = f(t, x), \qquad (' = d/dt) \tag{1.1}$$

and those of the perturbed systems

$$y' = f(t, y) + g(t, y, Ty). \tag{1.2}$$

Here x, y are elements of R^n, f and g are functions with values in R^n. Let I be the interval $0 \leq t < \infty$ and D be a region in R^n. We always assume that $f \in C[I \times D, R^n]$, that $f_x(t, x)$ exists and is continuous on $I \times D$, that $g \in C[I \times D \times D, R^n]$, and that T is a continuous operator such that T maps C [I, D] into C [I, D]. We use $x(t, t_0, x_0)$ to denote the solution of (1.1) passing through x_0 at $t = t_0$ and $y(t, t_0, y_0)$ to denote the solution of (1.2) passing through y_0 at $t = t_0$. The symbol $|\cdot|$ will be used to denote any convenient vector norm in R^n. It is known [8] that the matrix

$$(\partial/\partial x_0)[x(t, t_0, x_0)] = \Phi(t, t_0, x_0)$$

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(**) Nella seduta dell'8 maggio 1976.

exists, and satisfies the variational equation

$$z' = f_x [t, x(t, t_0, x_0)] z, \tag{1.3}$$

such that $\Phi(t_0, t_0, x_0) = E$ (identity matrix), and

$$(\partial/\partial t_0) [x(t, t_0, x_0)] = -\Phi(t, t_0, x_0) f(t_0, x_0).$$

As remarked by Pachpatte [2] we can impose on T various meanings. For example, if $g(t, y, z)$ is of the form

$$g(t, y, z) = F(t, y) + z$$

and if the operator T is defined by

$$Ty(t) = \int_{t_0}^{t} K(t, s, y(s)) \, ds, \qquad 0 \le t_0 \le s \le t < \infty,$$

then (1.2) yields an integrodifferential system

$$y'(t) = f(t, y(t)) + F(t, y(t)) + \int_{t_0}^{t} K(t, s, y(s)) \, ds.$$

If T is defined by

$$Ty(t) = y_t,$$

where the symbol y_t is as defined in [8, vol II], then (1.2) is reduced to a functional-differential equation.

In Section 2, we shall generalize a result of Talpalaru [3] for the case when $f(t, x) = A(t) x$ and $g(t, y, Ty) = F(t, y)$. In Section 3, we establish "generalized" asymptotic equivalence between systems (1.1) and (1.2), without explicitly introducing this concept.

2. Asymptotic relationship (Linear case)

In this section we consider the systems (1.1) and (1.2) respectively in the form

$$x' = A(t) x \tag{2.1}$$

and

$$y' = A(t) y + f(t, y) + g(t, y, Ty), \tag{2.2}$$

where A is an $n \times n$ matrix continuous for $0 \le t < \infty$, and extend Theorem 2.1 of Talpalaru [3] to systems (2.1) and (2.2). To that end we need the following lemma due to Hallam [6].

LEMMA 1. *Let there exist constants t_0 and $k > 0$ such that*

$$\left[\int_{t_0}^{t} |\beta(t)\,\gamma(t)\,P\gamma^{-1}(s)\,\Gamma(s)|^q\,ds\right]^{1/q} \le k \qquad \textit{for} \quad t \ge t_0\,, q > 1\,,$$

and suppose that

$$\int^{\infty} |\Gamma^{-1}(t)\,\beta^{-1}(t)|^{-q}\,dt = \infty\,,$$

where $\gamma(t)$, $\beta(t)$, $\Gamma(t)$ are $n \times n$ nonsingular and continuous matrices on I, *and* P *is a projection.*

Then

$$\lim_{t\to\infty} |\beta(t)\,\gamma(t)\,P| = 0.$$

THEOREM 1. *Let $\Delta(t)$ and $\Gamma(t)$ be continuous $n \times n$ matrices defined for $t \ge 0$ with $|\Delta(t) - \Gamma(t)| \to 0$ as $t \to \infty$, $|\Delta(t)| \le M$, for all $t \ge 0$, for some constant $M > 0$ and*

$$\int^{\infty} |\Gamma^{-1}(t)\,\Delta^{-1}(t)|^{-q}\,dt = \infty \qquad (q > 1).$$

Let $X(t)$ be a fundamental matrix of the system (2.1) *and suppose that*

(i) *there are two supplementary projections P_1 and P_2 and a positive constant k such that*

$$\left[\int_{t_0}^{t} |\Delta(t)\,X(t)\,P_1\,X^{-1}(s)\,\Gamma(s)|^q\,ds + \int_{t}^{\infty} |\Delta(t)\,X(t)\,P_2\,X^{-1}(s)\,\Gamma(s)\,ds|^q\,ds\right] \le k,$$

for all $t \ge t_0 \ge 0$,

(ii) *there is a nonnegative continuous function $\lambda(t)$ in $L_p[0, \infty)$, such that*

$$|\Delta^{-1}(t)|\,\Gamma^{-1}(s) f(s, y(s))| \le \lambda(s)\,|y(s)|\,, \qquad t, s \in I\,, |y| < \infty\,,$$

(iii) *there is a function $w(t, u, v) \in C[I \times R_+ \times R_+, R_+]$, $R_+ = [0, \infty)$, nondecreasing in u and v and with the property that $w(t, a, b) \in L_p[0, \infty)$ for each a and b, and furthermore*

$$|\Delta^{-1}(t)|\,|\Gamma^{-1}(s)\,g(s, y(s), Ty(s))| \le w(s, |y(s)|, |Ty(s)|), \quad t, s \in I\,, |y| < \infty,$$

(iv) $|Ty| \le k'\,|y|$ *for* $y \in D$.

Then for each bounded solution $x(t)$ of (2.1) *there is a bounded solution $y(t)$ of* (2.2) *such that*

$$(*) \qquad \lim_{t\to\infty} |\Delta(t)\,y(t) - \Gamma(t)\,x(t)| = 0\,.$$

Proof. Let t_0 be fixed arbitrarily, then for each bounded continuous function $y(t)$ defined on $R_{t_0} = [t_0, \infty)$ with values in R^n we define the norm $\|y\| = \sup_{t \geq t_0} |y(t)|$.

Let $x(t)$ be a bounded solution of (2.1) with $\|x\| \leq \rho/3$, $\rho > 0$, and let

$$B_\rho = \{u \in C[R_{t_0}, R^n] : \|u\| \leq \rho\}.$$

We define an operator τ on B_ρ as follows:

$$\tau y(t) = x(t) + \int_{t_0}^{t} X(t) P_1 X^{-1}(s) \{f(s, y(s)) + g(s, y(s), Ty(s))\} ds \tag{2.3}$$

$$- \int_{t}^{\infty} X(t) P_2 X^{-1}(s) \{f(s, y(s)) + g(s, y(s)), Ty(s))\} ds, \; t \geq t_0.$$

Using (ii), (iii) and (iv), and applying Hölder's inequality we obtain

$$|\tau y(t)| \leq \rho/3 + k\rho \left[\int_{t_0}^{t} (\lambda(s))^p ds\right]^{1/p} + k \left[\int_{t_0}^{t} \{w(s, \rho, k'\rho)\}^p ds\right]^{1/p} +$$

$$+ k\rho \left[\int_{t}^{\infty} (\lambda(s))^p ds\right]^{1/p} + k \left[\int_{t}^{\infty} \{w(s, \rho, k'\rho)\}^p ds\right]^{1/p}.$$

In view of the properties of λ and w it is possible to choose t_0 such that

$$\int_{t_0}^{\infty} (\lambda(s))^p ds \leq (1/6\,k)^p \qquad \text{and} \qquad \int_{t_0}^{\infty} \{w(s, \rho, k'\rho)\} ds \leq (\rho/6\,k)^p, \tag{2.4}$$

and consequently

$$|\tau y(t)| \leq \rho,$$

which shows that $\tau B_\rho \subset B_\rho$.

Next we show that τ is a continuous operator.

Suppose $y_n \in B_\rho$ and $y_n \to y$ uniformly on every compact subinterval of R_{t_0}. For each ε choose $t_1 \geq t_0$ such that

$$\rho \left[\int_{t_1}^{\infty} (\lambda(s))^p ds\right]^{1/p} + \left[\int_{t_1}^{\infty} \{w(s, \rho, k'\rho)\}^p ds\right]^{1/p} < \varepsilon/2k, \tag{2.5}$$

We let

$$\psi_i(t, s) = \Delta(t)\, x(t)\, P_i X^{-1}(s)\, \Gamma(s), \qquad i = 1, 2.$$

The for $t_0 \leq t \leq t_1$ we have:

$$|\tau y_n - \tau y(t)| \leq G_1 + G_2 + G_3,$$

where

$$G_1 = \int_{t_0}^{t_1} |\psi_1(t,s)|\,|\Delta^{-1}(t)|\,|\Gamma^{-1}(s)\{f(s,y_n(s)) - f(s,y(s))\}|\,ds +$$

$$+ \int_{t_0}^{t} |\psi_1(t,s)|\,|\Delta^{-1}(t)|\,|\Gamma^{-1}(s)\{g(s,y_n(s),Ty_n(s)) - g(s,y(s),Ty(s))\}|\,ds,$$

$$\leq \left[\int_{t_0}^{t_1} |\psi_1(t,s)|^q\,ds\right]\cdot\left\{\left[\int_{t_0}^{t} (|\Delta^{-1}(t)|\,|\Gamma^{-1}(s)(f(s,y_n(s)) - f(s,y(s)))|^p\,ds\right]^{1/p} +\right.$$

$$\left. + \left[\int_{t_0}^{t_1} (|\Delta^{-1}(t)|\,|\Gamma^{-1}(s)(g(s,y_n(s),Ty_n(s)) - g(s,y(s),Ty_n(s)))|^p\,ds\right]^{1/p}\right\} \leq$$

$$\leq k\left[\int_{t_0}^{t_1} (2\rho\lambda(s))^p\,ds\right]^{1/p} + k\left[\int_{t_0}^{t_1} (2w(s,\rho,k'\rho)^p\,ds\right]^{1/p},$$

and

$$G_3 = \int_{t_1}^{\infty} |\psi_2(t,s)|\,|\Delta^{-1}(t)|\,|\Gamma^{-1}(s)(f(s,y_n(s)) - f(s,y(s)))|\,ds$$

$$+ \int_{t_1}^{\infty} |\psi_2(t,s)|\,|\Delta^{-1}(t)|\,|\Gamma^{-1}(s)(g(s,y_n(s),Ty_n(s)) - g(s,y(s),Ty(s)))|\,ds \leq$$

$$\leq k\left[\int_{t_1}^{\infty} (2\rho\lambda(s))^p\,ds\right]^{1/p} + k\left[\int_{t_1}^{\infty} (2w(s,\rho,k'\rho))^p\,ds\right]^{1/p} < \varepsilon.$$

The expression G_2 is obtained from G_1 by replacing ψ_1 in G_1 by ψ_2. It follows that the integral G_i $(i = 1, 2, 3)$ exists for all values of t_1: Consequently $|\tau y_n(t) - \tau y(t)| \leq G_1 + G_2 + G_3$

$$\leq k\left[\int_{t_0}^{t_1} (|\Delta^{-1}(t)|\,|\Gamma^{-1}(s)\{f(s,y_n(s)) - f(s,y(s))\}|)^p\,ds\right]^{1/p}$$

$$+ k\left[\int_{t_0}^{t_1} (|\Delta^{-1}(t)|\,|\Gamma^{-1}(s)\{g(s,y_n(s),Ty_n(s)) -\right.$$

$$\left. - g(s,y(s),Ty(s))\}|)^p\,ds\right]^{1/p} + \varepsilon.$$

Since $y_n(t) \to y(t)$ uniformly on every compact interval $[t_0, t_1]$, it follows from the continuity of f, g, T and the fact that ε is arbitrary that

$$\tau y_n(t) \to \tau y(t) \qquad \text{as} \quad n \to \infty$$

uniformly on every compact subinterval of R_{t_0}. Hence τ is continuous. Using the fact that $x(t)$ is a solution of (2.1) one can easily verify that $\tau y(t)$ is a solution of the equation

$$v' = A(t)v + f(t, y(t)) + g(t, y(t), Ty(t)), \tag{2.6}$$

and hence the set $\{(\tau y)' : y \in B_\rho\}$ is uniformly bounded on every finite subinterval of R_{t_0}. Consequently τB is equicontinuous. One can apply Schauder's fixed point theorem to conclude that the operator equation $\tau r = y$ has a solution $y = y(t)$ which satisfies (2.6) and hence (2.2).

Now we demonstrate that (*) holds. We have

$$|\Delta(t)y(t) - \Gamma(t)x(t)| \leq |\Delta(t) - \Gamma(t)||x(t) + H_1 + H_2,$$

where

$$H_2 = \int_t^\infty |\Delta(t)X(t)P_2X^{-1}(s)f(s, y(s))| ds +$$

$$+ \int_t^\infty |\Delta(t)X(t)P_2X^{-1}(s)g(s, y(s), Ty(s))| ds \leq$$

$$\leq \int_t^\infty |\psi_2(t, s)||\Gamma^{-1}(s)f(s, y(s))| ds +$$

$$+ \int_t^\infty |\psi_2(t, s)||\Gamma^{-1}(s)g(s, y(s), Ty(s))| ds.$$

If we take the norm of $n \times n$ matrix A to be the sum of the absolute values of all the elements, then

$$n = |\Delta^{-1}(t)\Delta(t)| \leq |\Delta^{-1}(t)||\Delta(t)|, \qquad \text{for} \quad t \in [t_0, \infty),$$

and by hypothesis $|\Delta(t)| \leq M$, so $|\Delta^{-1}(t)| \geq n/|\Delta| \geq n/M$, consequently $1/|\Delta^{-1}(t)| \leq M/n$.

But we have

$$|\Delta^{-1}(t)||\Gamma^{-1}(s)f(s, y(s))| \leq \rho\lambda(s),$$

so

$$|\Gamma^{-1}(s)f(s, y(s))| \leq \rho\lambda(s)/|\Delta^{-1}(t)| \leq (M\rho/n)\lambda(s). \tag{2.7}$$

Similarly

$$|\Gamma^{-1}(s)\,g(s\,,y(s)\,,\mathrm{T}y(s))| \leq (\mathrm{M}/n)\,\omega(s\,,\rho\,,k'\rho)\,, \tag{2.8}$$

for $s \in [t_0\,,\infty)$. Consequently, using (2.7), (2.8) and Hölder's inequality we obtain

$$\mathrm{H}_2 \leq (k\mathrm{M}\rho/n)\left[\int_t^\infty (\lambda(s))^p\,\mathrm{d}s\right]^{1/p} + (k\mathrm{M}/n)\left[\int_t^\infty (\omega(s\,,\rho\,,k'\rho))^p\,\mathrm{d}s\right]^{1/p} \to 0$$

as $t \to \infty$. Now, for any $t_2 \geq t_0$,

$$\begin{aligned}
\mathrm{H}_1 &= \int_{t_0}^{t} |\Delta(t)\,\mathrm{X}(t)\,\mathrm{P}_1\,\mathrm{X}^{-1}(s) f(s\,,y(s))|\,\mathrm{d}s \\
&+ \int_{t_0}^{t} |\Delta(t)\,\mathrm{X}(t)\,\mathrm{P}_1\,\mathrm{X}^{-1}(s)\,g(s\,,y(s),\mathrm{T}y(s))|\,\mathrm{d}s \\
&= \int_{t_0}^{t_2} |\Delta(t)\,\mathrm{X}(t)\,\mathrm{P}_1\,\mathrm{X}^{-1}(s) f(s\,,y(s))|\,\mathrm{d}s \\
&= \int_{t_2}^{t} |\Delta(t)\,\mathrm{X}(t)\,\mathrm{P}_1\,\mathrm{X}^{-1}(s) f(s\,,y(s))|\,\mathrm{d}s \\
&+ \int_{t_0}^{t_2} |\Delta(t)\,\mathrm{X}(t)\,\mathrm{P}_1\,\mathrm{X}^{-1}(s)\,g(s\,,y(s)\,,\mathrm{T}y(s))|\,\mathrm{d}s \\
&+ \int_{t_2}^{t} |\Delta(t)\,\mathrm{X}(t)\,\mathrm{P}_1\,\mathrm{X}^{-1}(s)\,g(s\,,y(s)\,,\mathrm{T}y(s))|\,\mathrm{d}s.
\end{aligned}$$

Using (2.7), (2.8) and Hölder's inequality for the second and fourth term of the above equality, we obtain

$$\begin{aligned}
\mathrm{H}_1 \leq{}& (k\mathrm{M}\rho/n)\left[\int_{t_2}^{t} (\lambda(s))^p\,\mathrm{d}s\right]^{1/p} + (k\mathrm{M}/n)\left[\int_{t_2}^{t_2} (\omega(s\,,\rho\,,k'\rho))^p\,\mathrm{d}s\right]^{1/p} + \\
&+ \left[\int_{t_0}^{t_2} |\Delta(t)\,\mathrm{X}(t)\,\mathrm{P}_1|\,|\mathrm{X}^{-1}(s) f(s\,,y(s))|\,\mathrm{d}s\right. \\
&+ \left[\int_{t_0}^{t_2} |\Delta(t)\,\mathrm{X}(t)\,\mathrm{P}_1|\,|\mathrm{X}^{-1}(s)\,g(s\,,y(s)\,,\mathrm{T}y(s))|\,\mathrm{d}s\,.\right.
\end{aligned}$$

For each $\varepsilon > 0$, take $t_2 > t_0$ such that

$$\int_{t_2}^{\infty} (\lambda(s))^p \, ds \leq (n\varepsilon/4 k M \rho)^p \quad , \quad \int_{t_2}^{\infty} (\omega(s, \rho, k'\rho))^p \, ds \leq (n\varepsilon/4 k M)^p .$$

Since

$$\int_{t_0}^{t_2} | X^{-1}(s) f(s, y(s)) | \, ds \leq c ,$$

and

$$\int_{t_0}^{t_2} | X^{-1}(s) g(s, y(s), Ty(s)) | \, ds \leq c$$

where c is some positive real number, using Lemma 1, we can take t so large that

$$|\Delta(t) X(t) P_1| \left\{ \int_{t_0}^{t_2} (| X^{-1}(s) f(s, y(s)) | + | X^{-1}(s) g(s, y(s), Ty(s)) |) \, ds \right\} \leq \varepsilon/2 .$$

Consequently for sufficiently large t we will have $H_1 + H_2 \leq \varepsilon$. Since $| \Delta(t) - D(t) | \to 0$ as $t \to \infty$, and $| x(t) | \leq \rho$ for all $t \in R_{t_0}$, we obtain

$$| \Delta(t) y(t) - D(t) x(t) | \to 0 \qquad \text{as} \quad t \to \infty .$$

Thus the proof of Theorem 1 is complete.

We now deal with a converse problem to that considered in Theorem 1.

THEOREM 2. *Let the hypothesis of Theorem* 1 *hold. Then for each bounded solution* $y(t)$ *of* (2.2) *there is a bounded solution* $x(t)$ *of* (2.1) *such that* (*) *holds.*

Proof. Let $y(t)$ be a bounded solution of (2.2).

Define

$$\text{(2.9)} \quad x(t) = y(t) - \int_{t_0}^{t} X(t) P_1 X^{-1}(s) \{ f(s, y(s)) + g(s, y(s), Ty(s)) \} \, ds + $$
$$+ \int_{t}^{\infty} X(t) P_2 X^{-1}(s) \{ f(s, y(s)) + g(s, y(s), Ty(s)) \} \, ds ,$$

It is easy to verify that the integrals in (2.9) exist for $t \geq t_0$, and that $x(t)$ satisfies (2.1).

The rest of the proof follows that of Theorem 1.

Remarks. If $p = 1$, then $q = \infty$ then condition (i) becomes

$$\sup_{t_0 \leq s \leq t} | \Delta(t) X(t) P_1 X^{-1}(s) \Gamma(s) | + \sup_{t \leq s < \infty} | \Delta(t) X(t) P_2 X^{-1}(s) \Gamma(s) | \leq k .$$

Since Lemma 1 is not true in this case, it is necessary to assume that $\lim_{t\to\infty} | \Delta(t) X^{-1}(t) P_1 | = 0$.

2) Theorem 1 includes Theorem 2.1 of Talpalaru [3].

3) In case $\Delta(t)$ and $\Gamma(t)$ are constant matrices it should be possible to interpret the results geometrically. However, we do not intend to get involved in such a project.

3. Asymptotic relationship (nonlinear case)

In this section we establish an asymptotic relationship between the solutions of the nonlinear system

$$x' = f(t, x), \tag{3.1}$$

and its perturbed system

$$y' = f(t, y) + g(t, y, Ty). \tag{3.2}$$

We assume that for arbitrary $t_0 \geq \alpha \geq 0$ and $x_0 \in D$ the solution $x(t, t_0, x_0)$ of (3.1) exists for $\alpha \leq t \leq t_0$ and has values in D. This of course implies that the corresponding matrix $\Phi(t, t_0, x_0)$ exists in the same circumstances.

Now we state and prove the following theorem.

THEOREM 1. *In addition to condition* (iv) *assume that*

(v) $$\begin{aligned} &| \Phi(t, s, y(s)) g(s, y(s), Ty(s)) - \Phi(t, s, z(s)) g(s, z(s)\, Tz(s)) | \\ &\leq W(s, | T(y(s) - z(s)) |) \, | y(s) - z(s) |, \qquad t, s \in I, \end{aligned}$$

where $W(t, u) \in C[I \times R_+, R_+]$ *and is monotone non decreasing in* u *for each* $t \in I$,

(vi) $$| \Phi(t, s, 0) g(s, 0, 0) | \leq \mu(s), \qquad t, s \in I,$$

where μ *is a non-negative continuous function on* I *such that*

$$\int_\alpha^\infty \mu(s)\, ds < \infty \qquad \alpha \geq 0,$$

(vii) $$\int_{t'}^\infty W(s, a)\, ds \leq J(t'), \quad \textit{for each} \quad a \geq 0,$$

where

$$J(t') \to 0 \quad \textit{as} \quad t' \to \infty.$$

Then for each bounded solution $x(t)$ of (3.1) *there exists a bounded solution $y(t)$ of* (3.2) *such that*

$$\lim_{t\to\infty} |y(t) - x(t)| = 0 .$$

Remark 1. The proof of this theorem is unusually lengthy. The main idea is borrowed from the proof of a similar result by Marlin and Strubbe [1]. We will give below the outlines only.

Proof of the Theorem. Let $x(t)$ be a bounded solution of (3.1) with $\|x\| \leq \rho/2$. For $y \in B_\rho$ define τ as follows:

$$\tau y(t) = x(t) - \int_t^\infty \Phi(t, s, y(s)) g(s, y(s), Ty(s))\, ds , \qquad t \geq t_0 . \tag{3.3}$$

It is easy to show that τ is a well-defined and continuous mapping on B_ρ into B_ρ, and that $\overline{\tau B_\rho}$ (closure) is compact. Since B_ρ is a closed convex subset of a locally convex Banach space S of all bounded continuous R^n-valued functions defined on R_{t_0}, we can apply Tychonoff's fixed point theorem in order to prove that $\tau y = y$ has a solution in B_ρ. The rest of the proof follows that of Theorem 4 of Marlin and Strubbe [1].

Theorem 2 below, deals with a converse problem to that considered in Theorem 1 above.

THEOREM 2. *Let the hypotheses of Theorem* 1 *hold. Then for any bounded solution $y(t)$ of* (3.2) *there is a bounded solution $x(t)$ of* (3.1) *such that*

$$\lim_{t\to\infty} |y(t) - x(t)| = 0 .$$

Proof. Let $y(t)$ be a bounded solution of (3.2). The function $x(t)$ defined by

$$x(t) = y(t) + \int_t^\infty \Phi(t, s, y(s)) g(s, y(s), Ty(s))\, ds , \tag{3.4}$$

is well-defined, and $|y(t) - x(t)| \to 0$ as $t \to \infty$. We need only prove that $x(t)$ is a bounded solution of (3.1). We note that

$$(d/ds)\,[x(t, s, y(s))] = \Phi(t, s, y(s)) g(s, y(s), Ty(s)). \tag{3.5}$$

Using the chain rule, the definition of Φ and (3.5) we have

$$(d/ds)\,[f(t, s, y(s))] = f_x t, x(t, s, y(s)))\, \Phi(t, s, y(s)) g(s, y(s), Ty(s)). \tag{3.6}$$

The relations (3.4) and (3.5) yield

$$x(t) = y(t) + \lim_{N\to\infty} \int_t^N (d/ds)\,[x(t, s, y(s)]\,ds \tag{3.7}$$

$$= y(t) + \lim_{N\to\infty} x(t, N, y(N)) - y(t)$$

$$= \lim_{N\to\infty} x(t, N, y(N)).$$

Differentiating (3.4), and using (3.5) and (3.6) we get

$$x'(t) = f(t, y(t)) + \int_t^\infty (d/ds)\,[f(t, x(t, s, y(s))]\,ds =$$

$$= f(t, y(t)) + \lim_{N\to\infty} \int_t^N (d/ds)\,(f(t, x(t, s, y(s))]\,ds =$$

$$= \lim_{N\to\infty} f(t, x(t, N, y(N))) = f(t, x(t)).$$

Since $y(t)$ is bounded so is $x(t)$ and thus the proof of the theorem is complete

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