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Some fixed point theorems

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Analisi matematica. — *Some fixed point theorems.* Nota di BRIAN FISHER, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si ottengono condizioni per l'esistenza ed unicità di un punto fisso per applicazioni di uno spazio metrico completo in sé.

In a paper by Kannan [2], he proved the following theorem:

THEOREM 1. *If* T *is a mapping of the complete metric space* X *into itself, satisfying the inequality*

$$\rho(Tx, Ty) \leq c[\rho(x, Tx) + \rho(y, Ty)] \tag{1}$$

for all x, y *in* X, *where* $0 \leq c < \frac{1}{2}$, *then* T *has a unique fixed point* z.

The following theorem follows from Theorem 1:

THEOREM 2. *If* T *is a mapping of the complete metric space* X *into itself, satisfying the inequality*

$$\rho(Tx, Ty) \leq a\rho(x, Tx) + b\rho(y, Ty)$$

for all x, y *in* X, *where* $0 \leq a, b$ *and* $a + b < 1$, *then* T *has a unique fixed point* z.

Prof. It follows from the inequality

$$\rho(Tx, Ty) \leq a\rho(x, Tx) + b\rho(y, Ty)$$

that

$$\rho(Tx, Ty) = \rho(Ty, Tx) \leq a\rho(y, Ty) + b\rho(x, Tx)$$

and on adding these two inequalities it follows that

$$\rho(Tx, Ty) \leq c[\rho(x, Tx) + \rho(y, Ty)],$$

where $c = \frac{1}{2}(a + b)$ and $0 \leq c < \frac{1}{2}$. The result now follows from Theorem 1.

In the particular case $b = 0$, it follows that every point x in X is mapped onto the fixed point z, since

$$\rho(z, Tx) = \rho(Tz, Tx) \leq a\rho(z, Tz) = a\rho(z, z) = 0,$$

for all x in X.

However, if T is a mapping satisfying the inequality

$$\rho(Tx, Ty) \leq b \max\{\rho(x, Tx), \rho(y, Ty)\} \tag{2}$$

(*) Nella seduta dell'8 maggio 1976.

for all x, y in X, where $0 \leq b < 1$, then T does not necessarily map every point x in X onto a single point z. In fact, any mapping T which satisfies inequality (1) satisfies inequality (2) with $b = 2c$.

We will now prove the following theorem:

THEOREM 3. *If* T *is a mapping of the complete metric space* X *into itself, satisfying the inequality*

$$\rho(Tx, Ty) \leq b \max\{\rho(x, Tx), \rho(y, Ty)\}$$

for all x, y *in* X, *where* $0 \leq b < 1$, *then* T *has a unique fixed point* z.

Proof. For an arbitrary point x in X we have

$$\rho(T^n x, T^{n+1} x) \leq b \max\{\rho(T^{n-1} x, T^n x), \rho(T^n x, T^{n+1} x)\}.$$

Now either

$$\rho(T^{n-1} x, T^n x) \leq \rho(T^n x, T^{n+1} x)$$

for some n, or

$$\rho(T^n x, T^{n+1} x) < \rho(T^{n-1} x, T^n x)$$

for $n = 1, 2, \cdots$.

If

$$\rho(T^{n-1} x, T^n x) \leq \rho(T^n x, T^{n+1} x)$$

for some n, the it follows that for this n

$$\rho(T^n x, T^{n+1} x) \leq b\rho(T^n x, T^{n+1} x)$$

and so

$$\rho(T^n x, T^{n+1} x) = 0$$

since $b < 1$. Hence $T^n x = z$ is a fixed point.

Alternatively if

$$\rho(T^n x, T^{n+1} x) < \rho(T^{n-1} x, T^n x)$$

for $n = 1, 2, \cdots$, it follows that

$$\rho(T^n x, T^{n+1} x) < b^n \rho(x, Tx)$$

for $n = 1, 2, \cdots$. Thus

$$\rho(T^n x, T^{n+r} x) \leq \rho(T^n x, T^{n+1} x) + \cdots + \rho(T^{n+r-1} x, T^{n+r} x)$$

$$\leq (b^n + \cdots + b^{n+r-1}) \rho(x, Tx) \leq \frac{b^n}{1-b} \rho(x, Tx).$$

Since $b < 1$, it follows that $\{T^n x\}$ is a Cauchy sequence with a limit z.

We now have

$$\begin{aligned}\rho(z, Tz) &\leq \rho(z, T^n x) + \rho(T^n x, Tz) \\ &\leq \rho(z, T^n x) + b \max\{\rho(T^{n-1} x, T^n x), \rho(z, Tz)\}\end{aligned}$$

and on letting n tend to infinity we see that

$$\rho(z, Tz) \leq b\rho(z, Tz).$$

Since $b < 1$, it follows that

$$Tz = z$$

and so z is a fixed point.

Thus, it follows in either case that a fixed point z exists.

Now suppose that T has a second fixed point z'. Then

$$\rho(z, z') = \rho(Tz, Tz') \leq b \max\{\rho(z, Tz), \rho(z', Tz')\} = 0.$$

It follows that $z = z'$ and so the fixed point is unique. This completes the proof of the theorem.

In a recent paper, see [1], the following theorem was proved:

THEOREM 4. *If* T *is a continuous mapping of the compact metric space* X *into itself, satisfying the inequality*

$$\rho(Tx, Ty) < \tfrac{1}{2}[\rho(x, Tx) + \rho(y, Ty)]$$

for all distinct x, y *in* X, *then* T *has a unique fixed point* z.

The following theorem is an immediate consequence of this theorem:

THEOREM 5. *If* T *is a continuous mapping of the compact metric space* X *into itself, satisfying the inequality*

$$\rho(Tx, Ty) < b\rho(x, Tx) + (1 - b)\rho(y, Ty)$$

for all distinct x, y *in* X *where* $0 < b < 1$, *then* T *has a unique fixed point* z.

For this theorem to have any meaning when $b = 1$, we must stipulate that

$$\rho(Tx, Ty) \leq \rho(x, Tx)$$

for all distinct x, y in X, where the inequality is strict when $Tx \neq x$. It again follows that every point x in X is mapped onto the fixed point z.

We can however prove the following theorem:

THEOREM 6. *If* T *is a continuous mapping of the compact metric space* X *into itself, satisfying the inequality*

$$\rho(Tx, Ty) < \max\{\rho(x, Tx), \rho(y, Ty)\}$$

for all distinct x, y *in* X, *then* T *has a unique fixed point* z.

Proof. Define a function f on X by

$$f(x) = \rho(x, Tx)$$

for all x in X. Since ρ and T are continuous functions it follows that f is a continuous function on X. Since X is compact there exists a point z in X such that

$$f(z) = \inf\{f(x) : x \in X\}.$$

Assuming that $Tz \neq z$, we have

$$f(Tz) = \rho(Tz, T^2 z) < \max\{\rho(z, Tz), \rho(Tz, T^2 z)\} = \rho(z, Tz) = f(z),$$

giving a contradiction. It follows that z is in fact a fixed point of T.

If z' is a second fixed point of T then

$$\rho(z, z') = \rho(Tz, Tz') < \max\{\rho(z, Tz), \rho(z', Tz')\},$$

if $z \neq z'$, giving a contradiction. Thus $z = z'$ and so the fixed point is unique. This completes the proof of the theorem.

References

[1] B. Fisher - *A fixed point mapping*, « Bull. Calcutta Math. Soc. » (to appear).
[2] R. Kannan (1968) - *Some results on fixed points*, « Bull. Calcutta Math. Soc. », *60* 71-6.