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On Blumberg's theorem

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Analisi matematica. — *On Blumberg's theorem.* Nota di OFELIA TERESA ALAS, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si stabilisce un'estensione di un teorema di Blumberg includente altre più o meno recenti estensioni [4, 5, 2].

H. Blumberg showed that if f is a real-valued function on R^n, there is a dense subset D of R^n such that f restricted to D is continuous. H. R. Bennet [2], J. C. Bradford and Casper Goffman [4], H. E. White Jr. [5], extended this theorem in different ways. An extension of Blumberg's theorem (including the precedent ones) will be proved here.

First we recall some definitions and theorems.

Let X be a topological space, (Y, d) be a metric space and $f: X \to Y$ be a function.

DEFINITION 1 ([2]). The function f is said to approach $x \in X$ First Categorically (written $f\ 1 \to x$) if there is an $\varepsilon > 0$ and a neighborhood $N(x, \varepsilon)$ of x such that

$$M(x, \varepsilon) = \{z \in N(x, \varepsilon) \mid d(f(z), f(x)) < \varepsilon\}$$

is a First Category set.

DEFINITION 2 ([2]). The function f is said to approach $x \in X$ densely (written $f \to x$ densely) if given $\varepsilon > 0$ there is a neighborhood $N(x, \varepsilon)$ of x such that

$$M(x, \varepsilon) = \{z \in N(x, \varepsilon) \mid d(f(z), f(x)) < \varepsilon\}$$

is dense in $N(x, \varepsilon)$. If $D \subset X$ and x is a limit point of D then f is said to approach x densely via D (written $f \to x$ densely via D) if given $\varepsilon > 0$ there is a neighborhood $N(x, \varepsilon)$ of x such that $M(x, \varepsilon) \cap D$ is dense in $N(x, \varepsilon) \cap D$.

DEFINITION 3 ([2]). An open set $U \subset X$ is a partial neighborhood of a point $x \in X$ if either $x \in U$ or x is a limit point of U.

DEFINITION 4 ([5]). X has a σ-disjoint pseudo-base if there is a set $B = \cup \{B_n \mid n = 1, 2, \cdots\}$ of open subsets of X such that for each n the members of B_n are pairwise disjoint and for every nonempty open set $U \subset X$ there is a nonempty $V \in B$ contained in U.

THEOREM A ([2]). *If* $x \in X$, *then* $f \to x$ *densely if and only if for each partial neighborhood* U *of* x, $f(x)$ *is a limit point of* $f(U)$.

(*) Nella seduta dell'8 maggio 1976.

THEOREM B (Banach). *If* $E \subset X$ *is such that each point of* E *is First Category relative to* X *then* E *itself is of First Category in* X.

Following the proof of Theorem (1.6) of [2] we have

THEOREM 1. *Let* X *be a topological space,* (Y, d) *be a second countable metric space and* $f: X \to Y$ *be a function. Then* $F_1 = \{x \in X \mid f\ 1 \to x\}$ *and* $F_2 = \{x \in X \mid f$ *does not densely approach* $x\}$ *are sets of First Category in* X.

Proof. If $x \in F_1$ there is $\varepsilon(x) > 0$ and a neighborhood $N(x, \varepsilon(x))$ of x such that $M(x, \varepsilon(x)) = \{z \in N(x, \varepsilon(x)) \mid d(f(z), f(x)) < \varepsilon(x)\}$ is a set of First Category in X. (With no lost of generality we may assume $\varepsilon(x)$ of the form $1/m$ where $m = 1, 2, \cdots$). For each $k = 1, 2, \cdots$ let $C(k) = \{x \in F_1 \mid \varepsilon(x) = 1/k\}$ and let $D(k) = \{a(k, i) \mid i = 1, 2, \cdots\}$ be a countable dense subset of $f(C(k))$. Let $D = \cup \{a(k, i) \mid k, i = 1, 2, \cdots\}$. If $a(m, i) \in D$ let $R(m, i) = \{x \in C(m) \mid d(a(m, i), f(x)) < 1/2m\}$ and if $x \in R(m, i)$ let

$$RM(x, i) = \{z \in M(x, 1/m) \mid d(f(z), a(m, i)) < 1/2m).$$

Now, if $x, y \in R(m, i)$ and $z \in RM(x, i) \cap N(y, 1/m)$ then $z \in RM(y, i)$. Indeed, $x, y \in R(m, i)$ imply $d(a(m, i), f(x)) < 1/2m$ and $d(a(m, i), f(y)) < 1/2m$; $z \in RM(x, i)$ implies $z \in M(x, 1/m)$ and $d(f(z), a(m, i)) < 1/2m$. Thus, $d(f(z), f(y)) < 1/m$ and $z \in N(y, 1/m)$; it follows that $z \in M(y, 1/m)$ and $d(f(z), a(m, i)) < 1/2m$; in consequence $z \in RM(y, i)$. Putting $T(m, i) = \cup \{RM(x, i) \mid x \in R(m, i)\}$ we have that $T(m, i)$ is of First Category in each of its points and by Theorem B is of First Category in X. Finally, we have that $F_1 \subset \cup \{T(m, i) \mid m, i = 1, 2, \cdots\}$ and this last set is of First Category in X.

Let us now prove the second part of the theorem. For each $x \in F_2$ there is $\varepsilon(x) > 0$ such that for each neighborhood $N(x, \varepsilon(x))$ of x the set $M(x, \varepsilon(x)) = \{z \in N(x, \varepsilon(x)) \mid d(f(z), f(x)) < \varepsilon(x)\}$ is not dense in $N(x, \varepsilon(x))$.

Since Y is second countable let $\{G_n \mid n = 1, 2, \cdots\}$ be an open basis of Y. For each $n = 1, 2, \cdots$ let

$$F(n) = \{x \in F_2 \mid f(x) \in G_n \subset B(f(x), \varepsilon(x))\}.$$

Since F_2 is contained in $\cup \{F(n) \mid n = 1, 2, \cdots\}$ it is enough to prove that the interior of the closure of $F(n)$ is empty for each $n = 1, 2, \cdots$ On the contrary, let G be a nonempty open set contained in the closure of $F(n)$ for some n; if $p, q \in G \cap F(n)$ then $d(f(p), f(q)) < \varepsilon(p)$ and $\{q \in G \mid d(f(p), f(q)) < \varepsilon(p)\}$ would be dense in G which is not possible. The proof is completed.

THEOREM 2. *Let* X *be a Baire topological space,* (Y, d) *be a second countable metric space and* $f: X \to Y$ *be a function. There is a dense set* $D \subset X$ *such that if* $x \in D$ *then* $f \to x$ *densely via* D.

Proof. This theorem is a generalization of Theorem 2.2 of [2]. Let $F_1 = \{x \in X \mid f \,1 \to x\}$; by Theorem 1 F_1 is of First Category in X. Now put $X_1 = X - F_1$ and $F_2 = \{x \in X_1 \mid f$ does not densely approach x via $X_1\}$; by virtue of Theorem 1 F_2 is of First Category in X_1 (and thus in X). Put $D = X - (F_1 \cup F_2)$; D is dense in X and we shall prove that for each $x \in D$, $f \to x$ densely via D. Indeed, let $x \in D$ and let U be a partial neighborhood of x in X (thus, $U \cap D$ is a partial neighborhood of x in D). Since $x \notin F_2$, $f \to x$ densely via X_1; given $\varepsilon > 0$, there is a neighborhood $M(x, \varepsilon/2)$ of x in X such that

$$M(x, \varepsilon/2) \cap X_1 = \{z \in N(x, \varepsilon/2) \mid d(f(z), f(x)) < \varepsilon/2\} \cap X_1$$

is dense in $N(x, \varepsilon/2) \cap X_1$. Then, there is a $q \in U \cap M(x, \varepsilon/2) \cap X_1$ and, since q does not belong to F_1, the set $\{z \in U \mid d(f(z), f(q)) < \varepsilon/2\} \cap D$ is not of First Category in X. If y belongs to this last set, we have that $d(f(y), f(x)) < \varepsilon$; it follows that $f(x)$ is a limit point of $f(U \cap D)$, and by Theorem 1.5 of [2], $f \to x$ densely via D.

THEOREM 3. *Let* X *be a Baire semi-metrizable topological space,* (Y, d) *be a second countable metric space and* $f: X \to Y$ *be a function. There is a dense subset* D *of* X *such that* f *restricted to* D *is continuous.*

Proof. The proof follows in an analogous way of that of Theorem 2.5 of [2], since this last proof depends only on the existence of a dense subset D of X, such that for each $x \in D$, $f \to x$ densely via D.

THEOREM 4. *Let* X *be a Baire Hausdorff space with a* σ*-disjoint pseudo-base,* Y *be a Hausdorff second countable space and* $f: X \to Y$ *be a function. There is a dense subset* D *of* X *such that the restriction of* f *to* D *is continuous.*

Proof. Following Proposition 1.7 of [5], let $\mathscr{P} = \cup \{P_n \mid n = 1, 2, \cdots\}$ be a σ-disjoint pseudo-base of X; we may assume that, for each n, $G_n = \cup P_n$ is dense in X and P_n refines P_{n-1}. Since X is a Baire space, $X' = \cap \{G_n \mid n = 1, 2, \cdots\}$ is dense in X. Put $P(X') = \{P \cap X' \mid P \in \mathscr{P}\}$. Then $P(X')$ is a base for a topology τ^* on X' and is a pseudo-base for the subspace topology on X'. Since each element of $P(X')$ is open-closed in (X', τ^*), this last space is regular and $P(X')$ is a σ-discrete base for τ^*; thus, (X', τ^*) is pseudo-metrizable.

Furthermore, (X', τ^*) is a Baire space. On the other hand, since Y is a Hausdorff second countable space, there is a second countable metric space Z and a function $g: Z \to Y$, which is continuous and bijective.

Now we have the Baire pseudo-metrizable space (X', τ^*), the second countable metric space Z and the function $h: X' \to Z$, which assigns to each $x \in X'$ the element $z \in Z$, where $g(z) = f(x)$. Since every Baire pseudo-metrizable space contains a dense Baire metrizable subspace, we may apply Theorem 3 and there is a dense subset D of X' such that the restriction of h to D is continuous. But, since g is continuous, the restriction of f to D is continuous. (Indeed, for each $x \in D$, $f(x) = g(h(x))$).

We now give a example to show that the hypothesis of second countability of Y in Theorem 4 cannot be entirely avoided.

Example. Let R be the real line (with the usual topology) and Y be the discrete space over the real numbers. Let $f: R \to Y$ be the identity function. Let us assume that there exists a dense subset D of R such that the restriction of f to D is continuous. For each $y \in f(D)$, the inverse image set $f^{-1}(\{y\})$ is an open unitary set in D; thus, there is an open set U_y in R such that $f^{-1}(\{y\}) = U_y \cap D$. It follows that the open set U_y has just one point in D, which impossible, because D is dense in R.

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