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Kanhaya Lal Singh

Some applications of Kuratowskis measure of noncopactness

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Topologia. — *Some applications of Kuratowski's measure of noncopactness.* Nota di KANHAYA LAL SINGH, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — In Kuratowski [4] trovasi introdotta la nozione di misura della non compattezza; a questa si legano strettamente i concetti di k-insieme contrazione e di applicazione densificante dovuti rispettivamente a Darbo [2] ed a Furi e Vignoli [3]. Qui nel n. 1 si stabilisce un teorema sulla struttura dei punti fissi di un'applicazione densificante, estendente un risultato di Krasnoselskii e Sobolevskii [14]. Nel n. 2 si ottiene un teorema sulla alternativa di Fredholm per applicazioni densificanti, da cui si derivano quali corollari risultati di Petryshyn [17] e di Nussbaum. Infine, nel n. 3 si stabiliscono teoremi sul punto fisso relativi ad applicazioni densificanti, i quali generalizzano risultati di Vainberg [5] e di Hanani, Netanuahu e Reichaw–Reichbach [16].

INTRODUCTION

The notion of measure of noncompactness, which was introduced by K. Kuratowski [4], turns out to be the main tool for fixed point theory as well as in Integral and Differential equations [7], [8], [9], [10] and [11]. It was Darbo [2] who defined the concept of k-set contraction and proved the following fixed point Theorem (see Theorem D). Darbo's Theorem was extended by Furi and Vignoli [3] to densifying mappings. Without being aware of Furi and Vignoli's result Sadovskii [12] also extended the Theorem of Darbo [2], but using a different kind of measure of noncompactness (usually called the ball measure of noncompactness, see Definition 1.3). However these two measures of noncompactness share few properties in common (a counterexample for the case where they differ may be found in Nussbaum 13, pp. 127]). Nussbaum [13] using the measure of noncompactness of K. Kuratowski [4] developed the degree theory for k-set contraction with $k < 1$ and later extended it to densifying mappings. Since the densifying mappings are so general the generalization of classical fixed point theorems to such kind of mappings is of continuing interest.

In section 1 of the present paper we prove a theorem on the structure of fixed point set of densifying mappings, which generalizes the result of Krasnoselskii and Sobolevskii [14]. In section 2 we present a theorem on the Fredholm alternative for densifying mappings and obtain as a corollary the results of Petryshyn [17] and Nussbaum [13]. Finnally we prove some fixed point theorems for densifying mappings which generalize the results of M.M. Vainberg [5] and that of Hanani, Netanuahu and Reichaw-Reichbach [16].

(*) Nella seduta del 10 aprile 1976.

1. PRELIMINARY DEFINITIONS AND RESULTS

Let X be a Banach space. Let D be an open bounded subset of X. Let $\overline{D}$ and ∂D be respectively the closure and boundary of D.

DEFINITION 1.1 (Kuratowski). Let X be real Banach space and D be a bounded subset of X. The *measure of noncompactness* of D, denoted by $\gamma(D)$ is defined as follows:

$$\gamma(D) = \text{in } \{\varepsilon > 0/D \text{ can be covered by a finite number of subsets of diameter } < \varepsilon\}.$$

$\gamma(D)$ has the following properties:

(*a*) $0 \leq \gamma(D) \leq \delta(D)$, where $\delta(D)$ is the diameter of D,

(*b*) $\gamma(D) = 0$ if and only if D is precompact (i.e. $\overline{D}$ is compact),

(*c*) $C \subset D$ implies $\gamma(C) \leq \gamma(D)$,

(*d*) $\gamma(C \cup D) = \max\{\gamma(C), \gamma(D)\}$,

(*e*) $\gamma(C(D, r)) \leq \gamma(D) + 2r$, where $C(D, r) = \{x \text{ in } X/d(x, D) < r\}$,

(*f*) $\gamma(C + D) \leq \gamma(C) + \gamma(D)$, where $C + D = \{c + d/c \text{ in } C \text{ and } d \text{ in } D\}$.

DEFINITION 1.2 (Darbo). Let X be a real Banach space. Let $T : X \to X$ be a continuous mapping. T is said to be a *k-set contraction* if given any bounded but not precompact subset D of X we have

$$\gamma(T(D)) \leq k\gamma(D) \qquad \text{for some } k > 0.$$

In case $\gamma(T(D)) < \gamma(D)$, for any bounded subset D of X such that $\gamma(D) > 0$, then T is called *densifying mapping* (3). If $\gamma(T(D)) \leq \gamma(D)$, for any bounded subset D of X, then T si called *1-set contraction* [17].

DEFINITION 1.3 (Sadovskii). Let X be a real Banach space. Let $T : X \to X$ be a continuous mapping. T is said to be *condensing* if for any bounded subset of X with $\chi(D) > 0$ we have

$$\chi(T(D)) < \chi(D),$$

where $\chi(D)$ denotes the infimum of all real numbers $\varepsilon > 0$ such that D admits a finite ε-net.

DEFINITION 1.4. Let X and Y be two Banach spaces. Let D be a closed and convex subset of X. A mapping $T : D \to Y$ is said to be *compact* if it is continuous and maps bounded sets into relatively compact sets.

A mapping $T : D \to Y$ is said to be *completely continuous* if it takes each weakly convergent sequence into a strongly convergent sequence.

Remark 1.1. These two classes of mappings are not comparable. That is neither one is contained in the other. The counterexamples demonstrating the difference may be found in Vainberg [15, pp. 14–16]. In fact in Vainberg [15] these two mappings have been referred to as completely continuous and strongly continuous respectively.

DEFINITION 1.5. [20, pp. 30]. Let X and Y be two Banach spaces. A mapping $T : X \to Y$ is said to be a *completely continuous vectorfield* on X, if it can be represented as

$$T(x) = x - F(x), \tag{1}$$

where $F : X \to Z$, where Z is an arbitrary but fixed Banach space and F is a completely continuous mapping.

Remark 1.2. The sum of two k-set contractions is again a k-set contraction. Completely continuous and compact mappings are zero-set contractions. Contraction mappings (mappings of a closed, bounded and convex subset D of a Banach space X into itself satisfying the condition $\| T(x) - T(y) \| \le k \| x - y \|$, where $k < 1$, contractive mappings i.e. $\| T(x) - T(y) \| < \| x - y \|$, nonexpansive mappings i.e. $\| T(x) - T(y) \le \| x - y \|$) are respectively examples of k-set contraction with $k < 1$, densifying and 1-set contraction mappings.

THEOREM D (Darbo). *Let* C *be a bounded, closed and convex subset of a Banach space* X. *Let* $T : C \to X$ *be a k-set contraction with* $k < 1$. *Then* T *has a fixed point.*

DEFINITION 1.6. Let X and Y be two Banach spaces. Let $T : X \to Y$ be a densifying mapping. T is said to be *densifying vectorfield* on X, provided T can be expressed in the form

$$T(x) = x - F(x)$$

where $F : X \to Y$ is densifying.

DEFINITION 1.7. Let X be a real Banach space. Let D be a bounded subset of X. Let $T : \overline{D} \to X$ be a densifying mapping. We say T is *smoothable* on $\overline{D}$ if for each $\alpha > 0$ we can construct a densifying mapping T_α such that

$$\| T(x) - T_\alpha(x) \| < \alpha \qquad (x \text{ in } \overline{D})$$

and the equation

$$x = T_\alpha(x) + y$$

has a unique solution for y sufficiently small in norm. The mapping T_α will be called *smoothing*.

We know that if the degree of a densifying vectorfield $x - T(x)$ on the boundary ∂D of a bounded region D is different from zero, then the equation $x = T(x)$ in D has at least one solution.

Below we present a theorem characterizing the structure set of all solutions.

THEOREM 1.1. *Let* X *be a real Banach space. Let* D *be a bounded open subset of* X. *Let the degree of densifying vectorfield* $x - T(x)$ *be different from zero on* ∂D. *Let the densifying mapping* T *be smoothable on* $\overline{D}$. *Then the set of fixed points of the mapping* T *which lie in* D *is compact and connected.*

Proof. It follows clearly that A, the fixed point set of T is compact. Indeed, suppose A is not compact, then $\gamma(A) > 0$. But then $0 < \gamma(A) = \gamma(T(A)) < \gamma(A)$, a contradiction unless $\gamma(A) = 0$. Thus A is compact.

The main task remains to show that A is connected. Suppose A is not connected; then we can write A as

$$A = B \cup C$$

where B and C are nonempty, disjoint, closed sets; moreover $d(A, B) > 0$.

Let B_1 and C_1 ($B_1 \subset D$, $C_1 \subset D$) be disjoint neighbourhoods of B and C with respective boundaries ∂B_1 and ∂C_1. Let δ and η be the degrees of $x - T(x)$ on ∂B_1 and ∂C_1 respectively. Bet β be the degree of the densifying vectorfield $x - T(x)$ on ∂D.

Since the densifying vectorfield $x - T(x)$ does not vanish on $\overline{D} - (B + C)$ there exists an $r > 0$ such that

$$\| x - T(x) \| > r \qquad (x \text{ in } \overline{D} - (B + C)).$$

Let us first observe that the mapping $G(x)$ defined by

$$G(x) = x - T_\alpha(x) - T(x^*) + T_\alpha(x^*)$$

is a densifying vectorfield, where x^* is one of the fixed points of T and T_α is a smoothing mapping. Indeed, it is enough to show that $F(x) = T_\alpha(x) + T(x^*) - T_\alpha(x^*)$ is densifying Let M be any bounded but not precompact subset of D, then

$$\{F(M) + T_\alpha(x^*)\} \subset \{T_\alpha(M) + T(x^*)\}.$$

Therefore

$$\gamma\{F(M) + T_\alpha(x^*)\} \leq \gamma(F(M) + \gamma(x^*) < \gamma\{T_\alpha(M) + T(x^*)\} < \gamma(M) + \gamma(x^*)$$

since T_α and T are both densifying. Hence $\gamma(F(M)) < \gamma(M)$.

Therefore for sufficiently small α, the densifying vectorfield $x - T_\alpha(x) - T(x^*) + T_\alpha(x^*)$ will be homotopic on ∂D, ∂B_1 and ∂C_1 to the densifying vectorfield $x - T(x)$ by the following homotopy:

$$H(x, t) = x - t\,T(x) - (1 - t)(T_\alpha(x) + T(x^*) - T_\alpha(x^*)).$$

Indeed,

$$H(x, 0) = x - T_\alpha(x) - T(x^*) + T_\alpha(x^*) \quad \text{and} \quad H(x, 1) = x - T(x).$$

Moreover $x - T_\alpha(x) - T(x^*) + T_\alpha(x^*)$ does not vanish on $\overline{D} - (B_1 + C_1)$. Therefore the degrees $\beta(\alpha)$, $\gamma(\alpha)$ and $\eta(\alpha)$ of the densifying vectorfield $x - T_\alpha(x) - T(x^*) + T_\alpha(x^*)$ on ∂D, ∂B_1 and ∂C_1 will be equal.

Since by hypothesis the equation

$$x = T_\alpha(x) + T(x^*) - T_\alpha(x^*)$$

for sufficiently small α has a unique solution, the solution must be equal to x^*. Suppose x^* is in B_1: then the densifying vectorfield $x - T_\alpha(x) - T(x^*) + T_\alpha(x^*)$ has no zero vectors on ∂C_1, therefore the degree on ∂C_1 is equal to zero, that is $\eta = 0$. Similarly, assuming x^* in C_1 we can show that $\gamma = 0$. Thus we have $\beta = \eta + \gamma = 0$, a contradiction to our assumption. Thus the Theorem.

COROLLARY 1.1 (Krasnoselskii and Sobolevskii). *Let* E *be a real Banach space. Let* D *be a bounded region in* E. *Let* Γ *be the boundary of* D. *Let* $B : \overline{D} \to E$ *be a completely continuous operator. Let* γ, *the rotation of the field* $x - B(x)$, *be different from zero on* Γ *and let the completely continuous operator* B *be smoothable on* $\overline{D}$. *Then the set* T *of fixed points of the operator* B *which lie in* D *is compact and connected.*

2. DEFINITION 2.1. Let X and Y be two Banach spaces. Let $T : X \to Y$ be a bounded linear mapping. T is said to be *Fredholm* if

(1) $\alpha(T) = \dim N(T)$ is finite, where $N(T)$ denotes the null space of T,

(2) $\beta(T) = \dim N(T^*)$ is finite, where T^* is adjoint of T,

(3) $R(T)$, the range of T is closed in Y.

DEFINITION 2.2. Let $\Phi(X, Y)$ be the collection of Fredholm operators from X to Y. If T is in $\Phi(X, Y)$, then we define the *index* of T denoted by by $i(T)$ as follows:

$$i(T) = \alpha(T) - \beta(T).$$

THEOREM 2.2. *Let* X *be a Banach space. Let* $T_1, T_2 : X \to X$ *be bounded linear mappings. If* T_1 *is densifying and* T_2 *is* 1*-set contraction, then* $I - (T_1 \circ T_2)$, *where* I *is the identity mapping, is Fredholm of index zero.*

Proof. First we note that $T_1 \circ T_2$ is also a bounded linear mapping. Moreover $T_1 \circ T_2$ is densifying. Indeed, let D be any bounded but not precompact subset of X, then $(T_1 \circ T_2)(D) = T_1(T_2(D))$. Hence

$$\gamma\{(T_1 \circ T_2)(D)\} = \gamma\{T_1(T_2(D))\} < \gamma(T_2(D)) \leq \gamma(D).$$

Thus we need to show that $N(I - T_1 \circ T_2)$, the null space of $(I - T_1 \circ T_2) = \{x \text{ in } X / x - (T_1 \circ T_2)(x) = 0\}$, is finite dimensional. Recall that a space is finite dimensional if and only if the unit ball is compact in that space. Thus it suffices to show that $S = \{x \text{ in } N(I - T_1 \circ T_2) / \|x\| = 1\}$ is compact. Suppose not, i.e. S is not compact. Thus there exists a sequence $\{x_k\}$ contained in such that $\|x_n - x_m\| > 1/2$ for $n \neq m$ i.e. $\gamma\{(x_k)\} > 0$. Since $(T_1 \circ T_2)(x_k) = x_k$ for each k and $T_1 \circ T_2$ is densifying, therefore

$$0 < \gamma(\{x_k\}) = \gamma(\{(T_1 \circ T_2)(x_k)\} < \gamma(\{x_k\}),$$

a contradiction. Thus S is compact, so that $N(I - T_1 \cdot T_2)$ is finite dimensional.

Now we need to show that $R(I - T_1 \circ T_2)$, the range of $I - T_1 \circ T_2$, is closed. It is enough to show, by Theorem D (see below), that $I - T_1 \circ T_2$ maps any bounded closed set into itself, which is evident. Indeed, since $T_1 \circ T_2$ is densifying, by Lemma 1 [13, pp. 80] $I - T_1 \circ T_2$ is a closed map.

Thus $I - T_1 \cdot T_2$ is semi-Fredholm. Since for each λ in $[0, 1]$, $\lambda(T_1 \circ T_2)$ is densifying we conclude from the previous argument that $F_\lambda = I - \lambda(T_1 \circ T_2)$ is semi-Fredholm for each λ and therefore by the result [18, pp. 230] the index $i(F_\lambda)$ is continuous in λ. Since it is an integer (including possibly $\pm\infty$) it must be constant for $0 < \lambda < 1$, showing that $i(I - T_1 \circ T_2) = i(F_1) = i(F_0) = i(I) = 0$, i.e. $I - T_1 \circ T_2$ is a Fredholm of index zero.

Taking $T_2 = I$ we get the following results as corollaries:

COROLLARY 2.1. (Petryshyn). *Let* X *be a Banach space. Let* $T : X \to X$ *be a bounded, linear and densifying mapping. Then* $I - T$ *is Fredholm of of index zero.*

COROLLARY 2.2 (Nussbaum). *Let* X *be a Banach space. Let* $T : X \to X$ *be a bounded, linear and k-set contraction with* $k < 1$, *the* $I - T$ *is Fredholm of index zero.*

THEOREM D [19, pp. 489]. *Let* X *and* Y *be two Banach spaces. Let* $U : X \to Y$ *be a bounded and linear mapping. If* U *maps bounded and closed sets onto closed sets then* U *has a closed range.*

3. DEFINITION 3.1. Let X be a Banach space. Let X^* be its dual space. A mapping $T : X \to X^*$ is said to be a duality mapping if for every x in X we have

$$\| T(x) \| = \| x \| , \langle T(x), x \rangle = \| T(x) \| \| x \| = \| x \|^2 .$$

THEOREM 3.1. *Let* X *be a Banach space. Let* X^* *be its dual space. Let* $T : X \to X$ *be a densifying mapping. Let* $F : X \to X^*$ *be the duality mapping. If*

$$\langle T(x), F(x) \rangle \geq 0 \tag{1}$$

on some sphere $\| x \| = r > 0$, then there exists a vector x_0 in $D_r = \{x$ in $X / \| x \| = r\}$ such that $T(x_0) = 0$.

Proof. It is enough to show that the mapping $R = I - T$ has a fixed point in $\bar{D}_r$, where $I : X \to X$ is the identity mapping. Let us first note that R is clearly densifying. Now $\bar{D}_r$ is closed, bounded and convex, therefore by Dugundji's Theorem (20, pp. 25, Lemma 8.1) there exists a retraction V defined by

$$V(x) = \begin{cases} R(x) & \text{if } \| R(x) \| < r \\ r \cdot R(x) / \| R(x) \| & \text{if } \| R(x) \| \geq r. \end{cases}$$

Then clearly V is densifying. Indeed, let $f_1(x) = R(x), f_2(x) = 0$, $\lambda_1 = r / \| R(x) \|$ for $\| R(x) \| > r$ and $\lambda_1(x) = 1$ for $\| R(x) \| < r$ and $\lambda_2(x) = 1 - \lambda_1(x)$. Then by Proposition 9 [13, p. 17] $V(x)$ is densifying.

Moreover clearly V maps $\bar{D}_r$ into itself. Hence by a Theorem of Furi and Vignoli [3] we conclude the existence of a fixed point x_0 in $\bar{D}_r$ i.e. there exists x_0 in D_r such that $V(x_0) = x_0$.

Now we have following two cases:

Case 1. If $\| x_0 \| < r$, then $x_0 = V(x_0) = R(x_0) = x_0 - T(x_0)$ i.e. $T(x_0) = x_0$.

Case 2. If $\| x_0 \| = r$, then $x_0 = r \cdot R(x_0) / \| R(x_0) \|$, where $\| R(x_0) \| > r$, then

$$R(x_0) = x_0 \| R(x_0) \| / r = p x_0 , \qquad \text{where } p = \| R(x_0) \| / r > 1. \tag{2}$$

Now the assumption $p > 1$ leads to a contradiction, since by condition (1) we have

$$\langle F(x_0), R(x_0) \rangle = \langle F(x_0), x_0 - T(x_0) \rangle = \langle F(x_0), x_0 \rangle - \langle F(x_0), T(x_0) \rangle \leq \langle F(x_0), x_0 \rangle. \tag{3}$$

But from (2) we have

$$(4) \qquad \langle F(x_0), R(x_0)\rangle = \langle F(x_0), px_0\rangle = p\langle F(x_0), x_0\rangle .$$

From (3) and (4) we have

$$p\langle F(x_0), x_0\rangle \leq \langle F(x_0), x_0\rangle$$

or

$$(5) \qquad (p-1)\langle F(x_0), x_0\rangle \leq 0 .$$

Since F is a duality mapping, $\langle F(x_0), x_0\rangle = \| F(x_0)\| \| x_0 \| > 0$. Thus from (5) we conclude that $(p-1) \leq 0$, i.e. $p < 1$, but $p < 1$ contradicts (2), hence p must be equal to 1. Thus we have $R(x_0) = x_0 - T(x_0) = x_0$ i.e. $T(x_0) = 0$.

COROLLARY 3.1. *Let* H *be a Hilbert space. Let* $T: H \to H$ *be a densifying mapping. Suppose there exists* $R > 0$ *such that* $\langle T(x), x\rangle \leq \| x \|^2$ *for all* x, *with* $\| x \| = R$. *Then* T *has a fixed point in* H.

Proof. It is enough to show that condition (1) of Theorem 3.1. is satisfied. Since we are in a Hilbert space, H^* is again H. Now taking for the duality mapping F, the identity mapping I, condition (1) becomes $\langle T(x), x\rangle \geq 0$.

Define now the mapping $B = I - T$; then clearly B is densifying, moreover for all x with $\| x \| = R$ we have

$$\langle B(x), x\rangle = \langle x - T(x), x\rangle = \langle x, x\rangle - \langle x, T(x)\rangle = \| x \|^2 - \langle x, T(x)\rangle \geq 0 .$$

Therefore by Theorem 3.1. there exists an x_0 such that $B(x_0) = 0$ or equivalently $0 = x_0 - T(x_0)$, which in turns implies that T has a fixed point.

COROLLARY 3.2. [5, pp. 286, Lemma 22.3]. *Let* E_n *be a finite dimensional Banach space. Let* $U: E_n \to E_n$ *be a duality mapping. Let* $T: E_n \to E_n$ *be a continuous mapping. Then if*

$$\langle U(x), T(x)\rangle \geq 0$$

on some sphere $\| x \| = r > 0$, *then there exists a vector* x_0 *in* $D_r = \{x \text{ in } E_n / \| x \| \leq r\}$ *such that* $T(x) = 0$.

Proof. Since E_n is finite dimensional and T is continuous, T is compact, hence a 0-set contraction. Thus the Corollary 3.2. follows from Theorem 3.1.

DEFINITION 3.2. Let X be a real Banach space. Let X^* be the dual space of X. A mapping $T: X \to X^*$ is said to be *coercive* if

$$\langle T(x), x\rangle \geq c(\| x \|) \| x \| ,$$

where $c(t)$ is a real-valued function of nonnegative t such that $c(t) \to \infty$ as $t \to \infty$.

COROLLARY 3.3. *Let* X *be a reflexive Banach space and let* X^* *be its dual. Let* $T : X \to X^*$ *be a coercive densifying operator. Then the mapping* $T : X \to X^*$, *is surjective, i.e. the equation* $T(x) = y$ *has a solution for any vector* y *in* X^*.

Proof. Since the operator T is coercive, there exists a real-valued function $\alpha(t)$ of the nonnegative variable t such that $\alpha(t) \to \infty$ as $t \to \infty$ and

$$\langle T(x), x\rangle \geq \|x\| \, \alpha(\|x\|).$$

Consider now the mapping

$$F_y(x) = T(x) - y$$

where y is an arbitrary but fixed vector in X^*. Then F_y is densifying. Indeed, let D be any bounded but not precompact subset of X, then

$$\{F(D) + y\} \subseteq \{T(D)\}.$$

Therefore

$$\gamma\{F_y(D) + y\} < \gamma(F_y(D)) + \gamma(y) < \gamma(T(D)) < \gamma(D).$$

Since $\gamma(y) = 0$, $\gamma(F_y(D)) < \gamma(D)$. Thus F_y is densifying. Moreover

$$\langle F_y(x), x\rangle = \langle T(x) - y, x\rangle = \langle T(x), x\rangle - \langle x, y\rangle \geq \|x\| (\alpha \|x\| - \|y\|).$$

Therefore there exists a positive constant M_y such that $\langle F_y(x), x\rangle > 0$, whenever $\|x\| > M_y$. Thus the mapping F_y satisfies the conditions of Theorem 3.1. with duality mapping I. Hence we conclude the existence of a point x_0 such that $F_y(x_0) = 0 = T(x_0) - y$. Thus $T(x_0) = y$. Hence the Theorem.

In [23] we proved the following Theorem for densifying mapping using the condition (K) of A. Altman [20, pp. 66].

THEOREM K (Theorem 5.1. [23]). *Let* X *be a Banach space. Let* D *be an open ball about the origin in* X. *Let* $T : \overline{D} \to X$ *be a densifying mapping such that*

$$\text{(K)} \qquad \|x - T(x)\|^2 \geq \|T(x)\|^2 - \|x\|^2 \qquad x \text{ in } \partial D.$$

Then T *has a fixed point in* X.

If we translate the origin to the point x_0 the condition (K) can be stated as

$$\text{(K}^*\text{)} \qquad \|T(x) - x\|^2 > \|T(x) - x_0\|^2 - \|x - x_0\|^2 .$$

As Corollaries of Theorem K, we have the following Corollaries.

COROLLARY 3.4. *Let* X *be a Hilbert space. Let* $S = \{x \text{ in } X / \|x - x_0\| \leq r\}$ *be a sphere in* X. *Let* $T : S \to X$ *be a densifying mapping such that*

$$\text{(H)} \qquad \langle x - x_0, T(x) - x_0\rangle \leq r^2$$

for x *in* ∂S. *Then there exisists a point* z *in* S *such that* $T(z) = z$.

Proof. The Corollary will follow from Theorem K if we can show that condition (H) implies condition (K*).

Suppose (H) is satisfied bu not (K*), i.e.

(1) $$\langle T(x) - x_0 , x - x_0 \rangle \leq r^2$$

but $\| T(x) - x \|^2 \not\geq \| T(x) - x_0 \|^2 - \| x - x_0 \|^2$, i.e.

(2) $$\| T(x) - x \|^2 < \| T(x) - x_0 \|^2 - \| x - x_0 \|^2 .$$

Now

(3) $$\| T(x) - x \|^2 = \| T(x) - x_0 - (x - x_0) \|^2 =$$
$$= \langle T(x) - x_0 - (x - x_0) , T(x) - x_0 - (x - x_0) \rangle$$
$$\| T(x) - x_0 \|^2 - 2 \langle T(x) - x_0 , x - x_0 \rangle + 2 \| x - x_0 \|^2 .$$

From (2) and (3) we have

$$2 \langle T(x) - x_0 , x - x_0 \rangle > 2 \| x - x_0 \|^2 ,$$

or $$\langle T(x) - x_0 , x - x_0 \rangle > \| x - x_0 \|^2 = r^2 ,$$

a contradiction to (H).

COROLLARY 3.5. [16, Hanani, Netanuahu and Reichaw-Reichbach]. *Let* $T : S \to X$ *be a completely continuous mapping of a sphere* $S = \{x$ *in* $X / \| x - x_0 \| \leq r\}$ *in the Hilbert space* X *into* X *such that*

$$\langle x - x_0 , T(x) - x_0 \rangle \leq r^2$$

for x *in* ∂S. *Then there exists a point* z *in* S *such that* $T(z) = z$.

COROLLARY 3.6. *Let* $T : S \to S$ *be a densifying mapping of a sphere* $S = \{x$ *in* $X / \| x - x \| \leq r\}$ *in a Hilbert space* X *into* X, *such that for some* $\beta \neq 0$ *the mapping* $x + \beta T(x)$ *is densifying on* S. *Suppose further that for given some* y *in* X *the following condition is satisfied*

(J) $$\langle x - x_0 , \beta (T(x) - y) \rangle \leq 0$$

for all x *in* ∂S. *Then there exists a point* z *in* S *such that* $T(z) = y$.

Proof. We wish to show that condition (J) for the mapping $F(x) = x + \beta (T(x) - y)$ implies condition (H) of our Corollary 3.4. First let us observe that $F(x)$ is also densifying. It remains to show that (J) implies (H) for the mapping F. Indeed, for x in ∂S we have

$$\langle x - x_0 , F(x) - x_0 \rangle = \langle x - x_0 , x - x_0 + \beta (T(x) - y) \rangle =$$
$$= \| x - x_0 \|^2 + \langle x - x_0 , \beta (T(x) - y) \rangle = r^2 + \langle x - x_0 , \beta (T(x) - y) \rangle$$

since $\langle x - x_0 , \beta (T(x) - y) \rangle \leq 0$.

Hence

$$\langle x - x_0 , F(x) - x_0 \rangle \leq r^2.$$

Thus by Corollary 3.4. we conclude the existence of a point z in S such that $z = F(z) = z + \beta T(z) - \beta y$ or $\beta T(z) = \beta y$ which in turns implies that $T(z) = y$.

Taking $\beta = -1$ and $x_0 = 0$ in Corollary 3.6. we have the following

COROLLARY 3.7. *If* $T : S \to X$ *is a mapping of the sphere* $S = \{x \text{ in } X / \| x \| \leq r\}$ *in a Hilbert space* X *into itself such that for some given* y *in* X

$$\langle x - T(x) \rangle \geq \langle x , y \rangle$$

for $\| x \| = r$ *and* $x - T(x)$ *is densifying on* S. *Then there exists a point* z *in* S *such that* $T(z) = y$.

COROLLARY 3.8 [16, Hanani, Netanuahu and Reichaw-Reichbach]. *Let* $T : S \to S$ *be a mapping of a sphere* $S = \{x \text{ in } X / \| x - x_0 \| \leq r\}$ *in a Hilbert space* X *into* X, *such that for some* $\beta \neq 0$ *the mapping* $x + \beta T(x)$ *is completely continuous on* S. *Suppose further that for given* y *in* X *the following condition is satisfied*

$$\text{(J)} \qquad \langle x - x_0 , \beta (T(x) - y) \rangle \leq 0$$

for all x *in* ∂S. *Then there exists a point* z *in* S *such that* $T(z) = y$.

COROLLARY 3.9. [16, Hanani, Netenuahu and Reichaw-Reichbach]. *If* $T : S \to X$ *is a mapping of the sphere* $S = \{x \text{ in } X / \| x \| \leq r\}$ *in a Hilbert space* X *into itself such that for some given* y *in* X.

$$\langle x , T(x) \rangle \geq \langle x , y \rangle$$

for $\| x \| = r$ *and* $x - T(x)$ *is completely continuous in* S. *Then there exists a point* z *in* S *such that* $T(z) = y$.

REMARK 3.1. As concluding remark we note that all the results of [16] mentioned in the present paper follow as the Corollary of A. Altman's Theorem [20, pp. 66].

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