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Spaces of real Grassmannians

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Geometria algebrica. — *Spaces of real Grassmannians.* Nota di
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RIASSUNTO. — Se n, m, d sono interi soddisfacenti le $1 \leq m \leq d \leq n$, si dimostra che l'insieme $G_m(R^d, R^n)$ di tutte le Grassmanniane reali $G_m(R^d)$ giacenti in una data $G_m(R^n)$ è una varietà topologica compatta ed un CW–complesso, di dimensione $m(d+1-m)(n-d)$.

§ 1. INTRODUCTION

Let $G_m(R^d, R^n)$ be the set of all real Grassmannians $G_m(R^d)$ which are subspaces of a Grassmannian $G_m(R^n)$, $1 \leq m \leq d \leq n$. In § 2, we show that $G_m(R^d, R^n)$ is a compact Hausdorff space. In § 3, we prove that $G_m(R^d, R^n)$ is a topological manifold and a CW-complex of dimension $m(d+1-m)(n-d)$. Also one finds that for all $m \geq 1$, $G_m(R^m, R^{m+1})$ is homeomorphic to S^m, the m-dimensional standard sphere. Finally, one deduces that $G_m(R^m, R^n)$ is not smooth if the pair (m, n) satisfies

either $m \neq 1, 2, 4$ or 8 and $n \geq m+2$

or $m = 8$ and $n > m+2$.

Remark. In a similar way, one can consider the case of complex Grassmannians and one obtains that $G_m(C^d, C^n)$ is a compact topological manifold and a CW-complex of complex dimension $m(d+1-m)(n-d)$ for $1 \leq m \leq d \leq n$.

§ 2. GENERALIZED REAL GRASSMANN SPACES $G_m(R^d, R^n)$

DEFINITION 2.1. A generalized real Grassmann space, $G_m(R^d, R^n)$, $1 \leq m \leq d \leq n$, is the set of all real Grassmannians $G_m(R^d)$ which are subspaces of a Grassmannian $G_m(R^n)$.

From p. 323 of [3], the point

$$x = (x^1, \cdots, x^m) \in V_m(R^n) \cap (R_1^{n-m+1} \times \cdots \times R_m^{n-m+1})$$

(where

$$x^j = (\overbrace{0, \cdots, 0}^{m-j}, x_1^j, \cdots, x_{n-m+1}^j, \overbrace{0, \cdots, 0}^{j-1})$$

$$\equiv (0, \cdots, 0, x_1^j, \cdots, x_{n-d}^j, y_1^j, \cdots, y_{d+1-m}^j, 0, \cdots, 0)$$

$$R_j^{n-m+1} = \{(\overbrace{0, \cdots, 0}^{m-j}, \xi_1, \cdots, \xi_{n-m+1}, \overbrace{0, \cdots, 0}^{j-1})\}, \qquad j = 1, \cdots, m$$

(*) Nella seduta del 10 aprile 1976.

and $V_m(R^n)$ is the Stiefel manifold of m-frames in R^n) determines a point in $G_m(R^n)$ and every point of $G_m(R^n)$ is determined by such a point. In the same way, the space $G_m(R^d, R^n)$ is determined by the equations

$$y_i^j = \sum_{k=1}^{n-d} v_{ik}^j x_k^j \qquad (j = 1, \cdots, m\,;\, i = 1, \cdots, d+1-m).$$

Thus a point of $G_m(R^d, R^n)$ is determined by an array

$$(*) \qquad \begin{pmatrix} v_1^1 & \cdots & v_1^m \\ \cdots & \cdots & \cdots \\ v_{d+1-m}^1 & \cdots & v_{d+1-m}^m \end{pmatrix} \equiv (V_1, \cdots, V_m) \quad \text{or} \quad \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_{d+1-m} \end{pmatrix}$$

where

$$V_j \in V_{d+1-m}(R^n) \cap (R_{1j}^n \times \cdots \times R_{d+1-mj}^n), \qquad j = 1, \cdots, m$$

$$\omega_i \in V_m(R^n) \cap (R_{i1}^n \times \cdots \times R_{im}^n), \qquad i = 1, \cdots, d+1-m$$

and

$$R_j^{n-m+1} \supset R_{ij}^n = \{(\overbrace{0, \cdots, 0}^{m-j+i-1}, \xi_1, \cdots, \xi_{n-d+1}, \overbrace{0, \cdots, 0}^{d-m+j-i})\}$$

$$(1 \leq i \leq d+1-m\ ;\ 1 \leq j \leq m).$$

Also every point of $G_m(R^d, R^n)$ is determined by such an array. Note that

$$\begin{pmatrix} \lambda_1^1 v_1^1 & \cdots & \lambda_1^m v_1^m \\ \cdots & \cdots & \cdots \\ \lambda_{d+1-m}^1 v_{d+1-m}^1 & \cdots & \lambda_{d+1-m}^m v_{d+1-m}^m \end{pmatrix},$$

where all the λ are nonzero real numbers, determines the same point $G_m(R^d, R^n)$ as (*) above. An array which determines a point of $G_m(R^d, R^n)$ can then be considered as a member of the intersection $V_m(R^d, R^n) \subset A^m$ (where A represents the $(d+1-m)$-fold Cartesian product $R^n \times \cdots \times R^n$) of the Cartesian products

$$\prod_{j=1}^{m} [V_{d+1-m}(R^n) \cap (R_{1j}^n \times \cdots \times R_{d+1-mj}^n)]$$

and

$$\prod_{i=1}^{d+1-m} [V_m(R^n) \cap (R_{i1}^n \times \cdots \times R_{im}^n)].$$

We then have a canonical function

$$q : V_m(R^d, R^n) \to G_m(R^d, R^n)$$

which maps each array to the member of $G_m(R^d, R^n)$ which it determines. We give $G_m(R^d, R^n)$ the quotient topology.

Alternatively, let $V_{d+1-m}^0(R^n)$ be the subset of $V_{d+1-m}(R^n)$ consisting of all $(d+1-m)$-tuples of linearly independent unit vectors of R^n. Then

$G_m(R^d, R^n)$ can also be considered as an identification space of $V^0_m(R^d, R^n)$ which is the intersection of

$$\prod_{j=1}^{m} [V^0_{d+1-m}(R^n) \cap (R^n_{1j} \times \cdots \times R^n_{d+1-mj})]$$

and

$$\prod_{i=1}^{d+1-m} [V^0_m(R^n) \cap (R^n_{i1} \times \cdots \times R^n_{im})].$$

One sees from the following diagram that both constructions yield the same topology for $G_m(R^d, R^n)$.

$$\begin{array}{ccccc} V^0_m(R^d, R^n) \subset & V_m(R^d, R^n) & \xrightarrow{\;g\;} & V^0_m(R^d, R^n) \\ & {\scriptstyle q_0} \searrow & \downarrow {\scriptstyle q} & \swarrow {\scriptstyle q_0} \\ & & G_m(R^d, R^n) & \end{array}$$

Here g normalizes all the vectors in an array and q_0 denotes the restriction of q to $V^0_m(R^d, R^n)$.

LEMMA 2.2. *The space of Grassmannians* $G_m(R^d, R^n)$ *is a compact Hausdorff space. Furthermore, the correspondence* $X \rightarrow X^{\perp}$ *which assigns to each* X *in* $G_m(R^d, R^n)$ *its complement* $X^{\perp}$ *in* $G_m(R^n)$ *defines a homeomorphism between* $G_m(R^d, R^n)$ *and* $G_m(R^{n+m-d-1}, R^n)$.

Proof. The set $V^0_m(R^d, R^n)$ is a closed bounded subset of A^m, and therefore is compact. It follows that $G_m(R^d, R^n) = q_0(V^0_m(R^d, R^n))$ is also compact. Consider the following diagram:

$$\begin{array}{ccc} G_m(R^d, R^n) \subset & \overbrace{G_m(R^n) \times \cdots \times G_m(R^n)}^{d+1-m} \\ {\scriptstyle f^0_\omega} \searrow & \downarrow {\scriptstyle f_\omega} \\ & R \end{array}$$

Here f_ω is defined as

$$f_\omega(X_1, \cdots, X_{d+1-m}) = \rho_\omega(X_1) \cdots \rho_\omega(X_{d+1-m}),$$

where $\omega \in R^n$ and $\rho_\omega : G_m(R^d) \rightarrow R$ is a continuous function (see the proof of Lemma 5.1. in [4] for the definition of ρ_ω). Thus f_ω is continuous and so

f^0_ω which is the restriction of f to $G_m(R^d, R^n)$ is also continuous. Now if X, Y are distinct members of $G_m(R^d, R^n)$ such that

$$X = (X_1, \cdots, X_{d+1-m}) \quad \text{and} \quad Y = (Y_1, \cdots, Y_{d+1-m}),$$

and if $\omega \in X_k$ for some k and $\omega \notin Y_k$ for all k, then $f^0_\omega(X) \neq f^0_\omega(Y)$. This proves that $G_m(R^d, R^n)$ is Hausdorff space.

Finally define a function

$$f : V_m(R^d, R^n) \to V_m(R^{n+m-d-1}, R^n)$$

as follows. For each $(V_1, \cdots, V_m) \in V_m(R^d, R^n)$ which determines $X \in G_m(R^d, R^n)$ (where $V_j = (v^j_1, \cdots, v^j_{d+1-m}), j = 1, \cdots, m$) complete the basis $\{v^j_1, \cdots, v^j_{d+1-m}\}$ so that

$$(v^j_1, \cdots, v^j_{d+1-m}, u^j_1, \cdots, u^j_{n-d}) \in V_{n-m+1}(R^{n-m+1}_j), \qquad j = 1, \cdots, m.$$

Put

$$U_j = (u^j_1, \cdots, u^j_{n-d}), \qquad i = 1, \cdots, m.$$

Then $(U_1, \cdots, U_m) \in V_m(R^{n+m-d-1}, R^n)$ determines $X^\perp$. Now set

$$f(V_1, \cdots, V_m) = (U_1, \cdots, U_m)$$

and it follows that the following diagram is commutative.

$$\begin{array}{ccc} V_m(R^d, R^n) & \xrightarrow{f} & V_m(R^{n+m-d-1}, R^n) \\ \downarrow q & & \downarrow q \\ G_m(R^d, R^n) & \xrightarrow{\perp} & G_m(R^{n+m-d-1}, R^n) \end{array}$$

f is continuous implies qf is continuous. Thus the correspondence $X \mapsto X^\perp$ is also continuous. Also $\perp$ is one-to-one. Using the same argument one constructs an inverse function

$$\perp' : G_m(R^{n+m-d-1}, R^n) \to G_m(R^d, R^n)$$

which is also continuous and one-to-one. Thus f is a homeomorphism.

§ 3. A CELL STRUCTURE FOR $G_m(R^d, R^n)$

Let $G_m(R^n)$ be represented by points

$$x = (x^1, \cdots, x^m) \in V_m(R^n) \cap (R^{n-m+1}_1 \times \cdots \times R^{n-m+1}_m),$$

where

$$x^j = (\overbrace{0, \cdots, 0}^{m-j}, x^j_1, \cdots, x^j_{n-m+1}, \overbrace{0, \cdots, 0}^{j-1}), \qquad 1 \leq j \leq m.$$

The space $G_m(R^n)$ contains subspaces

$$G_m(R^m) \subset (G_m(R^{m+1}) \subset \cdots \subset G_m(R^n)$$

where $G_m(R^{m-1+k})$ is represented by points $x = (x^1, \cdots, x^m)$ where

$$x^j = (\overbrace{0, \cdots, 0}^{m-j}, x_1^j, \cdots, x_k^j, \overbrace{0, \cdots, 0}^{n+j-m-k}).$$

By a *generalized Schubert symbol* $\sigma^m = (a_m, \cdots, a_d)$ is meant a sequence of $d + 1 - m$ integers satisfying

$$m \leq a_m < a_{m+1} < \cdots < a_d \leq n.$$

For each symbol σ^m, let $e(\sigma^m) \subset G_m(R^d, R^n)$ denote the set of all Grassmannians X such that

$$\dim (X \cap G_m(R^{a_{m+i}})) = mi$$

and

$$\dim (X \cap G_m(R^{a_{m+i-1}})) = m(i-1), \qquad i = 1, \cdots, d-m.$$

Let $H_j^{k-m+1} \subset R_j^{n-m+1}$ $(1 \leq j \leq m)$ denote the set of all unit vectors in R^n of the form

$$(\overbrace{0, \cdots, 0}^{m-j}, \xi_1, \cdots, \xi_{k-m+1}, 0, \cdots, 0).$$

Also let $H_{ij}^n \subset R_{ij}^n$, $(1 \leq i \leq d + 1 - m ; 1 \leq j \leq m)$ denote the set of all unit vectors in R^n of the form

$$(\overbrace{0, \cdots, 0}^{m-j+i-1}, \xi_1, \cdots, \xi_{n-d+1}, \overbrace{0, \cdots, 0}^{d-m+j-i}) \qquad \text{with} \quad \xi_1 > 0.$$

LEMMA 3.1. *Each Grassmannian* $X \in e(\sigma^m)$ *is determined uniquely by an array*

$$\begin{pmatrix} v_1^1 & \cdots & v_1^m \\ \cdots & \cdots & \cdots \\ v_{d+1-m}^1 & \cdots & v_{d+1-m}^m \end{pmatrix}$$

where $v_i^j \in H_{ij}^n \cap H_j^{a_{m-1+i}-m+1}$ *for all* i, j.

Proof. $(v_1^1, \cdots, v_1^m)$ determines the m-plane

$$X \cap G_m(R^{a_m}).$$

Now v_1^j is required to lie in a 1-dimensional vector subspace of $H_{1j}^n \cap H^{a_m - m+1}$ and to be a unit vector. The condition that its $(m - j + 1)$-th coordinate be positive defines v_1^j uniquely, $j = 1, \cdots, m$. Note that

$$(v_1^1, \cdots, v_1^m) \in V_m^0(R^n).$$

Next $(v_2^1, \cdots, v_2^m)$ is required to determine an m-plane in the m-dimensional space

$$X \cap G_m(R^{a_{m+1}})$$

such that v_1^j and v_2^j are linearly independent and the v_2^j are also unit vectors. This implies that v_2^j is a unit vector in a 1-dimensional subspace of $H_{2j}^n \cap H_j^{a_{m+1}-m+1}$. Again since its $(m-j+2)$-th coordinate is supposed to be positive, this defines v_2^j uniquely, $j = 1, \cdots, m$. Also

$$(v_2^1, \cdots, v_2^m) \in V_m^0(R^n)$$

and v_1^j and v_2^j are linearly independent, $j = 1, \cdots, m$. Continuing by induction, one obtains a unique array for X.

LEMMA 3.2. *Let*

$$e'(\sigma^m) = \prod_{i,j} H_{ij}^n \cap \prod_{k=1}^{d+1-m} (H_1^{a_{m-1+k}-m+1} \times \cdots \times H_m^{a_{m-1+k}-m+1}).$$

Then $e'(\sigma^m) \subset V_m^0(R^d, R^n)$ *is topologically an open cell of dimension*

$$m \sum_{k=1}^{d+1-m} (a_{m+k-1} - k - m + 1).$$

Furthermore q *maps* $e'(\sigma^m)$ *homeomorphically onto* $e(\sigma^m)$.

Proof. The proof will be by induction on d. When $d = m$ we have that $\sigma^m = (a_m)$ and

$$e'(a_m) = \prod_{j=1}^{m} (H_{1j}^n \cap H_j^{a_m-m+1}) \cong \prod_{j=1}^{m} \text{Int}\,(D^{a_m-m})$$

where $\text{Int}\,(D^{a_m-m})$ denotes the interior of an $(a_m - m)$-dimensional disk. Thus in this case $e'(a_m)$ is topologically an open cell of dimension

$$m(a_m - m)$$

and the lemma is true in this case.

Now assume, as an inductive hypothesis, that the lemma is true for all cases of $d' < d$. We shall now prove it for $d' = d$.

$$e'(\sigma^m) = e'(a_m, \cdots, a_d) =$$

$$= \prod_{i=1}^{d+1-m} \left[\prod_{j=1}^{m} H_{ij}^n \cap (H_1^{a_{m-1+i}-m+1} \times \cdots \times H_m^{a_{m-1+i}-m+1}) \right]$$

$$= e'(a_m, \cdots, a_{d-1}) \times \left[\prod_{j=1}^{m} H_{d+1-mj}^n \cap (H_1^{a_d-m+1} \times \cdots \times H_m^{a_d-m+1}) \right]$$

$$= e'(a_m, \cdots, a_{d-1}) \times \left[\prod_{j=1}^{m} (H_{d+1-mj}^n \cap H_j^{a_d-m+1}) \right]$$

$$\cong e'(a_m, \cdots, a_{d-1}) \times \prod_{j=1}^{m} \text{int}\,(D^{a_d-d}).$$

Thus by the inductive hypothesis $e'(\sigma^m)$ is topologically isomorphic to the Cartesian product of two open cells of dimensions

$$m \sum_{k=1}^{d-m} (a_{m+k-1} - k - m + 1) \qquad \text{and} \qquad m(a_d - d)$$

and this proves the first part of the lemma.

By Lemma 3.1 q maps $e'(\sigma^m)$ in a one-to-one correspondence onto $e(\sigma^m)$ and since the spaces involved are Hausdorff, then q is a homeomorphism.

COROLLARY 3.3. *The generalized Grassmann space* $G_m(R^d, R^n)$ *is a topological manifold of dimension*

$$m(d + 1 - m)(n - d).$$

Proof. The proof follows easily from Lemma 2.2 and Lemma 3.2 since

$$G_m(R^d, R^n) = e(n - d + m, \cdots, n).$$

THEOREM 3.4. *The* $\binom{n - m + 1}{d - m + 1}$ *sets* $e(\sigma^m)$ *form the cells of a* CW-*complex with underlying space* $G_m(R^d, R^n)$. *Similarly taking the direct limit as* $n \to \infty$, *one obtains an infinite* CW-*complex with underlying space* $G_m(R^d, R^\infty)$.

Proof. We first show that each member in the boundary of a cell $e(\sigma^m)$ belongs to a cell of lower dimension. Let $\bar{e}'(\sigma^m)$ be the closure of $e'(\sigma^m)$ and let $\bar{e}(\sigma^m)$ also be the closure of $e(\sigma^m)$. Since $\bar{e}'(\sigma^m)$ is compact, the image of $\bar{e}'(\sigma^m)$ is equal to $\bar{e}(\sigma^m)$. Hence every X in the boundary $\bar{e}(\sigma^m) - e(\sigma^m)$ is determined by an array

$$\begin{pmatrix} v_1^1 & \cdots & v_1^m \\ \cdots & \cdots & \cdots \\ v_{d+1-m}^1 & \cdots & v_{d+1-m}^m \end{pmatrix}$$

which belongs to $\bar{e}'(\sigma^m) - e'(\sigma^m)$. Now

$$(v_i^1, \cdots, v_i^m) \in V_m(R^{a_{m-1+i}}), \qquad 1 \le i \le d + 1 - m.$$

Thus

$$\dim (X \cap G_m(R^{a_{m-1+i}})) \ge i$$

for each i and we have that the generalized Schubert symbol $\tau^m = (b_m, \cdots, b_d)$ associated with X must satisfy

$$b_m < b_{m+1} < \cdots < b_d.$$

Hence $\dim(e(\tau^m)) < \dim(e(\sigma^m))$. Together with Lemma 3.2, this completes the proof that $G_m(R^d, R^n)$ is a finite CW-complex.

Similarly $G_m(R^d, R^\infty)$ with the direct limit topology is a CW-complex for the same values of m and d. The closure finiteness condition is satisfied since each $X \in G_m(R^d, R^\infty)$ belongs to a finite subcomplex $G_m(R^d, R^n)$.

COROLLARY 3.5. *For any* m, *the infinite generalized Grassmannian* $G_m(R^m, R^\infty)$ *is a* CW-*complex having one* mr-*cell* $e(r + m)$ *for each integer* $r \ge 0$. *The closure* $\bar{e}(r + m) \subset G_m(R^m, R^\infty)$ *is equal to* $G_m(R^m, R^{m+r})$.

Proof. The proof follows immediately from the theorem.

COROLLARY 3.6. *The number of mr-cells in* $G_m(R^d, R^n)$ *is equal to the number of partitions of r into at most* $d+1-m$ *integers each of which is* $\leq n-d$.

Proof. To every generalized Schubert symbol $\sigma^m = (a_m, \cdots, a_d)$ with $\dim(e(\sigma^m)) = mr$, there corresponds a partition $i_1, \cdots, i_s$ of r where $i_1, \cdots, i_s$ denotes the sequence obtained from $a_m - m, \cdots, a_d - d$ by cancelling any zeros which may appear at the beginning of this sequence. Clearly

$$1 \leq i_1 \leq i_2 \leq \cdots \leq i_s \leq n-d$$

and $s = d+1-m$.

COROLLARY 3.7. *For any* $m \geq 1$, *the generalized Grassmann space* $G_m(R^m, R^{m+1})$ *is homeomorphic to* S^m, the *m-dimensional standard sphere.*

Proof. The cells of $G_m(R^m, R^{m+1})$ are $e(m)$, the 0-cell and $e(m+1)$ the m — cell. Also

$$G_m(R^m, R^{m+1}) = \bar{e}(m+1).$$

Clearly $\bar{e}(m+1) = e(m+1) \cup \text{point} \simeq S^m$.

COROLLARY 3.8. *The generalized Grassmann space* $G_m(R^m, R^n)$ *is not smooth if the pair* (m, n) *satisfies*

either	$m \neq 1, 2, 4$ or 8	and $n \geq m+2$
or	$m = 8$	and $n > m+2$.

Proof. Suppose $G_m(R^m, R^n)$ is smooth for the above values of m and n. Then using the cell structure of Corollary 3.5, the Duality Theorem (see e.g. Theorem 11.10 of [4]) and induction on n, one shows that the mod 2 cohomology ring $H^*(G_m(R^m, R^n); Z_2)$ of $G_m(R^m, R^n)$ is isomorphic to $Z_2[a]$ subject to the relation

$$a^{n-m+1} = 0$$

where $a \in H^m(G_m(R^m, R^n); Z_2)$. This fact, however, contradicts the theo rems of Adams [1] and Adem [2] (see for instance the remark on top of p. 134 of [4]).

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