Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Seiichi Yamaguchi, Tyuzi Adati

On holornorphically subprojective Kählerian manifolds, II

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **60** (1976), n.4, p. 405–413.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1976_8_60_4_405_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1976.

Geometria differenziale. — *On holomorphically subprojective Kählerian manifolds*, II. Nota di SEIICHI YAMAGUCHI e TYUZI ADATI, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — La nozione di varietà olomorficamente sottoproiettiva è stata introdotta dagli Autori in un altro lavoro [6]. Qui si definiscono e studiano quelle fra tali varietà che risultano kähleriane di 1^a o di 2^a specie.

1. INTRODUCTION

An n-dimensional affinely connected manifold A_n is said to be *subprojective* if there exists a coordinate system such that every geodesic is given with respect to this system by $n-2$ homogeneous linear equations and one further equation that need not be linear.

The present authors have introduced in a previous paper [6] the notion of holomorphically subprojective Kählerian manifold as follows. Let us consider an n complex dimensional Kählerian manifold M. If there exists a complex coordinate system such that every holomorphically planar curve is given with respect to this system by $n-2$ homogeneous linear equations and one further equation that need not be linear, then M is called *a holomorphically subprojective Kählerian manifold.*

In this paper, we investigate the differential geometric properties of the holomorphically subprojective Kählerian manifolds of the first and second kind. In § 2 we shall recall the identities and theorems in a Kählerian manifold with vanishing Bochner curvature tensor and, in § 3, we give a short summary of holomorphically subprojective Kählerian manifolds which are necessary for what follows. Moreover the holomorphically subprojective Kählerian manifolds of the first and second kind are defined. The holomorphically subprojective Kählerian manifold of the first kind is studied in § 4.

2. KÄHLERIAN MANIFOLDS WITH VANISHING BOCHNER CURVATURE TENSOR

Let us consider a $2n$ real dimensional Kählerian manifold M with complex structure J and Riemannian metric g which satisfy the following relations

$$J_j^{\ r} J_r^{\ i} = -\delta_j^i \quad , \quad g_{ji} = J_j^{\ r} J_i^{\ s} g_{rs} \quad , \quad \nabla_j J_i^{\ h} = 0 \, ,$$
$$(h, i, j, \cdots, r, s, t, \cdots = 1, 2, 3, \cdots, 2n)$$

∇ being the operator of covariant derivation with respect to the Riemannian connection defined by g. Denote by $R_{kji}{}^h$, R_{kj} and R the Riemannian curvature tensor, the Ricci tensor and the scalar curvature respectively.

(*) Nella seduta del 10 aprile 1976.

The tensor field defined by

$$K_{kji}{}^{h} = R_{kji}{}^{h} + (L_{ki}\,\delta_j^h - L_{ji}\,\delta_k^h + g_{ki}\,L_j^h - g_{ji}\,L_k^h \tag{2.1}$$
$$+ M_{ki}\,J_j^h - M_{ji}\,J_k^h + J_{ki}\,M_j^h - J_{ji}\,M_k^h + 2\,M_{kj}\,J_i^h + 2\,J_{kj}\,M_i^h)/[2\,(n+2)]$$

is called *the Bochner curvature tensor*, where we have put

$$L_{ji} = R_{ji} - R\,g_{ji}/[4\,(n+1)]\,M_{ji} = J_j^r\,L_{ri}\,. \tag{2.2}$$

In a Kählerian manifold M with vanishing Bochenr curvature tensor, the following equation

$$4\,(n+1)\,\nabla_k R_{ji} = g_{ki}\,\nabla_i R + g_{kj}\,\nabla_j R + 2\,g_{ji}\,\nabla_k R \tag{2.3}$$
$$- J_{kj}\,J_i^r\,\nabla_r R - J_{ki}\,J_j^r\,\nabla_r R$$

has been obtained by M. Matsumoto [3]. Furthermore M. Matsumoto and S. Tanno [4] have proved

THEOREM A. *If a Kählerian manifold* M *with vanishing Bochner curvature tensor has constant scalar curvature, then* M *is one of the following manifolds*:

(1) M *is a manifold of constant holomorphic sectional curvature.*

(2) M *is a locally product manifold of two manifolds of constant holomorphic sectional curvature* H ($\geqq 0$) *and* $-$H.

3. Holomorphically subprojective Kählerian manifolds

Now, in what follows, we always agree to adopt the following convention: (ij) (resp. $[ij]$) for indices i and j means the symmetric (resp. shew-symmetric) part with respect to indices i and j, for example

$$u_{(ij)} = u_{ij} + u_{ji} \qquad \text{(resp. } u_{[ij]} = u_{ij} - u_{ji}\text{)}.$$

As for a holomorphically subprojective Kählerian manifold, we have obtained the following in [6]:

THEOREM B. *In order that a* $2\,n\,(n \geqq 3)$ *real dimensional Kählerian manifold be holomorphically subprojective, it is necessary and sufficient that there exists a local real coordinate system* (x^i) *such that the Christoffel symbol of* M *may take the form*

$$\left\{ {h \atop ij} \right\} = \rho_{(i}\,\delta_{j)}^h + \tilde{\rho}_{(i}\,J_{j)}^h + f_{ji}\,x^h - f_{jr}\,J_i^r\,\hat{x}^h, \tag{3.1}$$

$$f_{[ji]} = 0 \quad , \quad f_{r[j}\,J_{i]}^r = 0\,, \tag{3.2}$$

where ρ_i *and* f_{ji} *are covariant vector and tensor fields respectively, and we have put* $\tilde{x}^h = J_r^{\ h} x^r$ *and* $\tilde{\rho}_h = - J_h^{\ r} \rho_r$.

Now we put

$$(3.3) \qquad Q_{kji} = \partial_{[k} f_{j]i} + x^r f_{r[k} f_{j]i} - \tilde{x}^r f_{r[k} f_{j]s} J_i^{\ s}, \qquad \partial_k f_{ji} = \frac{\partial f_{ji}}{\partial x^k},$$

$$(3.4) \qquad \lambda\, |x|^2 = Q_{kr}^{\ \ \ r} x^k, \ \varepsilon\, |x|^4 = Q_{kji}\, \tilde{x}^k x^j \tilde{x}^i, \ |x|^2 = g_{ji} x^j x^i,$$

$$(3.5) \qquad H_{ji} = \partial_i \rho_j - \rho_i \rho_j + \tilde{\rho}_i \tilde{\rho}_j + F_{ji}, \qquad F_{ji} = -(k + \rho_r x^r) f_{ji} + \rho_r \tilde{x}^r f_{js} J_i^{\ s},$$

$$(3.6) \qquad a\, |x|^2 = H_{ji} x^j x^i .$$

As for these quantities, the following relations have been introduced in [6]:

$$(3.7) \qquad \partial_j \rho_i = \partial_i \rho_j ,$$

$$(3.8) \qquad F_r^{\ r} = 0 ,$$

$$(3.9) \qquad 2(n-1) H_{ji} = \{2(n-1)a + (\lambda+\varepsilon)\, |x|^2\} g_{ji} - (\lambda+\varepsilon)(x_j x_i) + \tilde{x}_j \tilde{x}_i),$$

$$(3.10) \qquad 2(n-1)\, |x|^2 Q_{kji} + (\lambda+\varepsilon)\, |x|^2 (J_{i[k} \tilde{x}_{j]} + 2 \tilde{x}_i J_{jk} + g_{i[k} x_{j]}) + 2\{2\lambda + (n+1)\varepsilon\} \tilde{x}_i \tilde{x}_{[j} x_{k]} = 0 ,$$

$$(3.11) \qquad R_{kjih} = \left\{a + \frac{\lambda+\varepsilon}{2(n-1)} |x|^2\right\} (g_{i[k} g_{j]h} - J_{i[k} J_{j]h} + 2 J_{kj} J_{ih})$$
$$- \frac{\lambda+\varepsilon}{2(n-1)} [(J_{i[k} \tilde{x}_{j]} + 2 J_{jk} \tilde{x}_i - g_{i[j} x_{k]}) x_h - (J_{i[k} x_{j]} + 2 J_{jk} x_i$$
$$+ g_{i[j} \tilde{x}_{k]}) \tilde{x}_h + x_i (x_{[k} g_{i]h} - \tilde{x}_{[k} J_{j]h}) + \tilde{x}_i (\tilde{x}_{[k} g_{j]h} + x_{[k} J_{j]h}) - 2 J_{jh} \tilde{x}_{[k} x_{j]}]$$
$$- \frac{2\lambda + (n+1)\varepsilon}{(n-1)\, |x|^2} x_{[k} \tilde{x}_{j]} \tilde{x}_{[i} x_{h]} .$$

Next we give the identities of holomorphically subprojective Kählerian manifolds which are necessary for what follows.

From (3.11) we can find

$$(3.12) \qquad R_{kh} = -\left\{2(n+1)a + \frac{n(\lambda+\varepsilon)\, |x|^2}{n-1}\right\} g_{kh} + \frac{n\lambda+\varepsilon}{n-1} (x_k x_h + \tilde{x}_k \tilde{x}_h),$$

$$(3.13) \qquad R = -4n(n+1)a - 2\{n\lambda + (n+1)\varepsilon\}\, |x|^2,$$

from which, with the aid of these, it follows from (2.2) that

$$(3.14) \qquad L_{kh} = -\left\{(n+2)\,a + \frac{n\,(n+3)\,\lambda + (n+1)^2\,\varepsilon}{2\,(n-1)\,(n-1)}\,|\,x\,|^2\right\} g_{kh} + \frac{n\lambda+\varepsilon}{n-1}\,(x_k\,x_h + \tilde{x}_k\,\tilde{x}_h)\,.$$

Making use of (3.1), we have

$$(3.15) \qquad \nabla_j\,x_i = (1+\rho_r\,x^r)\,g_{ji} - \rho_r\,\tilde{x}^r\,J_{ji} + (\rho_j + f_{jr}\,x^r)\,x_i + (\tilde{\rho}_j + f_{jr}\,\tilde{x}^r)\,\tilde{x}_i\,,$$

or, equivalently

$$(3.15)' \qquad \nabla_j\,\tilde{x}_i = (1+\rho_r\,x^r)\,J_{ji} + \rho_r\,\tilde{x}^r\,g_{ji} + (\rho_j + f_{jr}\,x^r)\,\tilde{x}_i - (\tilde{\rho}_j - f_{jr}\,\tilde{x}^r)\,x_i\,.$$

Also, (3.5) and (3.9) yield the following

$$(3.16) \qquad \nabla_j\,\rho_i = -\rho_j\,\rho_i + \tilde{\rho}_j\,\rho_i + f_{ji} + [\{2\,(n-1)\,a + (\lambda+\varepsilon)\,|\,x\,|^2\}\,g_{ji} - (\lambda+\varepsilon)\,(x_j\,x_i + \tilde{x}_j\,\tilde{x}_i)]/[2\,(n-1)]$$

by virtue of (3.1).

We try to express the curvature tensor of a holomorphically subprojective Kählerian manifold M by means of the Ricci tensor.

We set $M' = \{P \in M : n\lambda + \varepsilon = 0 \text{ at } P\}$. Then in $M - M'$ the following relation is obtained by straightforward computations:

$$(3.17) \qquad R_{kjih} = (A + 2\,BD/E)\,(g_{k[i}\,g_{j]h} - J_{k[i}\,J_{j]h} + 2\,J_{kj}\,J_{ih}) - B\,(g_{k[i}\,R_{j]h} + R_{k[i}\,g_{j]h} + J_{k[i}\,S_{j]h} + S_{k[i}\,J_{j]h} + 2\,J_{kj}\,S_{ih} + 2\,S_{kj}\,J_{ih})/E + C\,[S_{kj}\,S_{ih} - D\,(S_{kj}\,J_{ih} + J_{kj}\,S_{ih}) + D^2\,J_{kj}\,J_{ih}]/E^2,$$

where we put

$$S_{kj} = J_k^{\ r}\,R_{rj} \quad , \quad A = a + \frac{(\lambda+\varepsilon)\,|\,x\,|^2}{2\,(n-1)} \quad , \quad B = \frac{\lambda+\varepsilon}{2\,(n-1)} \ ,$$

$$C = \frac{2\,\lambda + (n+1)\,\varepsilon}{(n-1)\,|\,x\,|^2} \ , \ D = -\{2\,(n+1)\,a + 2\,nB\,|\,x\,|^2\} \ , \ E = \frac{n\lambda+\varepsilon}{n-1}\ .$$

Thus we have

THEOREM 3.1. *In* $M - M'$, *the curvature tensor of the holomorphically subprojective Mählerian manifold* M *has the form* (3.17).

In order to get further results on holomorphically subprojective Kählerian manifolds, we consider the holomorphically subprojective Kählerian manifolds specified by the functions λ and ε.

In the following we shall call a holomorphically subprojective Kählerian manifold with

$$2\,\lambda + (n+1)\,\varepsilon = 0 \qquad (\text{resp. } n\lambda + \varepsilon = 0)$$

holomorphically subprojective Kählerian manifold of the first (resp. *second*) for brevity, we denore by M_1 (resp. M_2) the first (resp. second) kind.

In [6] we have derived the following

THEOREM C. M_1 $(n \geqq 3)$ *is a Kählerian manifold with vanishing Bochner curvature tensor which has the Ricci tensor of the form*

$$(3.18) \qquad R_{ji} = -\{2(n+1)a - n\varepsilon\,|x|^2/2\}\,g_{ji} - (n+2)\,\varepsilon\,(x_j x_i + \tilde{x}_j \tilde{x}_i)/2.$$

As for M_2, from (3.12) we have

THEOREM 3.2. *In order that a* $2n$ $(n \geqq 3)$ *dimensional holomorphically subprojective Kählerian manifold* M *be* M_2, *it is recessary and sufficient that* M *is an Einstein manifold.*

4. HOLOMORPHICALLX SUBPROJECTIVE KÄHLERIAN MANIFOLDS OF THE FIRST KIND

In the first place, let us seek the necessary and sufficient condition that a holomorphically subprojective Kählerian manifold M be M_1.

Let us derive the following

THEOREM 4.1. *In order that a* $2n$ $(n \geqq 3)$ *dimensional holomorphically subprojective Kählerian manifold* M *be the holomorphically subprojective Kählerian manifold of the first kind, it is necessary and sufficient that* M *has a vanishing Bochner curvature tensor.*

Proof. Transvecting (2.1) with $\tilde{x}^k x^j x^i \tilde{x}_h$ and according to our assumption, that is, $K_{kji}{}^h = 0$, it follows that

$$(4.1) \qquad R_{kijh}\,\tilde{x}^k x^j x^i \tilde{x}^h - 4\,|x|^2\,L_{kj}\,x^k x^j/(n+2) = 0\,.$$

By contraction (3.11) with $\tilde{x}^k x^j x^i \tilde{x}^h$, we get

$$R_{kjih}\,\tilde{x}^k x^j x^i \tilde{x}^h = -(4a + |x|^2)\,|x|^4$$

and in the same way we obtain from (3.14)

$$L_{kj}\,x^k x^j = -(n+2)\,a\,|x|^2 + \frac{n\lambda - (n+1)\,\varepsilon}{2\,(n+1)}\,|x|^4.$$

Consequently we can see that (4.1) denotes $2\lambda + (n+1)\,\varepsilon = 0$ by means of these. This concludes the proof.

THEOREM 4.2. M_1 $(n \geqq 3)$ *is a Kählerian manifold with vanishing Bochner curvature tensor which has the Ricci tensor of the form*

$$(4.2) \qquad R_{kj} = \alpha g_{kj} + \beta\,(\sigma_k \sigma_j + \tilde{\sigma}_k \tilde{\sigma}_j),$$

where σ_i is a gradient vector, and α and β are functions such that

$$\alpha = -2(n+1)a - n|\sigma|^2\beta/(n+2). \tag{4.3}$$

Proof. For some functions $\varkappa$ and μ we set

$$\sigma_i = \varkappa x_i - \mu \tilde{x}_i , \tag{4.4}$$

which yields that

$$x_i = \frac{\varkappa\sigma_i + \mu\tilde{\sigma}_i}{\varkappa^2+\mu^2} \quad , \quad \tilde{x}_i = \frac{\varkappa\tilde{\sigma}_i - \mu\sigma_i}{\varkappa^2+\mu^2} \quad , \quad |x|^2 = \frac{|\sigma|^2}{\varkappa^2+\mu^2} \quad ,$$

and substituting these into (3.18), it follows that

$$R_{ji} = -\left\{2(n+1)a - \frac{n\varepsilon|\sigma|^2}{2(\varkappa^2+\mu^2)}\right\} g_{ji} - \frac{(n+2)\varepsilon}{2(\varkappa^2+\mu^2)}(\sigma_j\sigma_i + \tilde{\sigma}_j\tilde{\sigma}_i) ,$$

or, equivalently,

$$R_{ij} = \alpha g_{ji} + \beta(\sigma_j\sigma_i + \tilde{\sigma}_j\tilde{\sigma}_i) ,$$

where we have put $\beta = -(n+2)\varepsilon/2(\varkappa^2+\mu^2)$ and α is given by (4.3). In what follows, we show that the covariant vector σ_i is gradient. In order to try this, we differentiate (4.4) covariantly. Then

$$\nabla_j\sigma_i = \varkappa_j x_i + \varkappa\nabla_j x_i - \mu_j\tilde{x}_i - \mu\nabla_j\tilde{x}_i ,$$

which means that

$$\nabla_{[j}\sigma_{i]} = \varkappa_{[j}x_{i]} + \varkappa\nabla_{[j}x_{i]} - \mu_{[j}\tilde{x}_{j]} - \mu\nabla_{[j}\tilde{x}_{i]} ,$$

where we have put $\varkappa_i = \nabla_i\varkappa$ and $\mu_i = \nabla_i\mu$. With the aid of (3.15) and (3.15)$'$, the equation above can be rewritten as follows:

$$\begin{aligned}\nabla_{[j}\sigma_{i]} = {} & \varkappa_{[j}x_{i]} - \mu_{[j}\tilde{x}_{i]} - 2\{\varkappa\rho_r\tilde{x}^r + \mu(1+\rho_r x^r)\} J_{ji} \\ & + \{\varkappa(\rho_{[j} + x^r f_{r[j}) + \mu(\tilde{\rho}_{[j} - \tilde{x}^r f_{r[j})\} x_{i]} \\ & + \{\varkappa(\tilde{\rho}_{[j} - \tilde{x}^r f_{r[j}) - \mu(\rho_{[j} + x^r f_{r[j})\} \tilde{x}_{i]} .\end{aligned} \tag{4.5}$$

Hereafter we may take $\varkappa$ and μ such that

$$\varkappa\omega = 1 + \rho_r x^r \quad , \quad \mu\omega = -\rho_r\tilde{x}^r \tag{4.6}$$

for certain function ω, because of $\varkappa$ and μ are any functions. Differentiating (4.6) covariantly and recalting (3.15), (3.15)$'$, (3.16), (4.4) and (4.6), equation (4.5) can be reduced to

$$\nabla_{[j}\sigma_{i]} = \sigma_i[2(\rho_j + f_{jr}x^r) - \partial_j\log\omega] - \sigma_j[2(\rho_i + f_{ir}x^r) - \partial_i\log\omega]$$

and therefore we have $d\sigma$ if there exists a function ω such that $\partial_j \log \omega = 2(\rho_j + f_{jr} x^r)$. In fact, there exists such a function ω. Let us prove this fact. Contracting h and i in (3.1), it follows that

$$\partial_j \log \sqrt{\mathfrak{g}} = 2(n+1)\rho_j + 2 f_{jr} x^r,$$

where $\mathfrak{g} = \det(g_{jk})$. Since our discussions are local, by virtue of (3.7) we can know that there exists a function ρ such that $\rho_i = \partial_i \rho$. So if we choose a function ω such that $\omega = \exp(\log \sqrt{\mathfrak{g}} - 2n\rho)$, then we find $\partial_i \log \omega = 2(\rho_i + f_{ir} x^r)$, which means that the proof is completed.

Let us prove the following

THEOREM 4.3. *In order that a* $2n$ $(n \geqq 3)$ *dimensional holomorphically subprojective Kählerian manifold* M *be a manifold of constant holomorphic sectional curvature, it is necessary and sufficient that* M *is of the first and of the second kind at the same time.*

Proof. If M is M_1 and M_2, then we have $\lambda = \varepsilon = 0$. Thus by making use of the Corollary in [6], we can see that M is a manifold of constant holomorphic sectional curvature. Conversely, we assume that M be a manifold of constant holomorphic sectional curvature. Then M has a vanishing Bochner curvature tensor and M is an Einstein manifold. Therefore by virtue of Theorem 3.2 and 4.1, we find that M is M_1 and M_2 at the same time.

In the next place, we try to show the following

THEOREM 4.4. *The holomorphically subprojective Kählerian manifold of the first kind with* $1 + \rho_r x^r = 0$ *and* $\rho_r \tilde{x}^r = 0$ *is one of following manifolds*:

(1) M *is a manifold of constant holomorphic sectional curvature.*

(2) M *is a locally product manifold of two manifolds of constant holomorphic sectional curvature* H $(\geqq 0)$ *and* $-$H.

Proof. By assumption we have

$$1 + \rho_r x^r = 0, \tag{4.7}$$

$$\rho_r \tilde{x}^r = 0. \tag{4.8}$$

We differentiate (4.7) covariantly and take account (3.15), (3.16), (4.7) and (4.8). Then we get $a = 0$ and hence equation (3.18) can be rewritten as follows:

$$\mathrm{R}_{ji} = \varepsilon\,[n\,|x|^2 g_{ji} - (n+2)(x_j x_i + \tilde{x}_j \tilde{x}_i)]/2. \tag{4.9}$$

which implies that

$$R = (n+1)(n-1)\,|x|^2\,\varepsilon\,, \tag{4.10}$$

and differentiating (4.9) and (4.10) covariantly and recalling (3.15), (3.15)', (4.7) and (4.8), we have

$$\nabla_k R_{ji} = \alpha_k\,[n\,|x|^2 g_{ji} - (n+2)(x_j x_i + \tilde{x}_j \tilde{x}_i)], \tag{4.11}$$

$$\nabla_k R = 2(n+1)(n-2)\,|x|^2\,\alpha_k\,, \tag{4.12}$$

where we have put

$$\alpha_k = \nabla_k \varepsilon/2 + \varepsilon\,(\rho_k + f_{kr}\,x^r).$$

Since our manifold has a vanishing Bochner curvature tensor by Theorem 4.3, we may use equation (2.3). By substitution of (4.11) and (4.12) into (2.3), we obtain

$$2\alpha_k\,[n\,|x|^2 g_{ji} - (n+2)(x_j x_i + \tilde{x}_j \tilde{x}_i)]$$
$$= (n-2)\,|x|^2\,(\alpha_j g_{ki} + \alpha_i g_{kj} + 2\,\alpha_k g_{ji} + J_{kj}\,\tilde{\alpha}_i + J_{ki}\,\tilde{\alpha}_j)\,,$$

from which we have by transvection with $x^j x^i$

$$n\,|x|^2\,\alpha_k = -(n-2)(\alpha_r x^r x_k + \alpha_r \tilde{x}^r \tilde{x}_k). \tag{4.13}$$

Furthermore, contracting this with x^k and $\tilde{x}^k$ respecvtiely, we can see that $\alpha_r x^r = \alpha_r \tilde{x}^r = 0$. Therefore from (4.13) we have $\alpha_k = 0$ together with these, which means that the scalar curvature R is constant. At last, the proof is completed by Theorem A.

Summing up the above discussions we can state the following

THEOREM 4.5. *A $2n$ ($n \geqq 3$) dimensional holomorphically subprojective Kählerian manifold of the first kind has the following properties (I) $\sim$ (III):*

(I) *The manifold is a Kählerian manifold with vanishing Bochner curvature tensor which has the Ricci tensor of the form*

$$R_{jk} = \alpha g_{jk} + \beta\,(\sigma_j \sigma_k + \tilde{\sigma}_j \tilde{\sigma}_k),$$

where σ_i is a gradient vector, and α and β are some functions.

(II) *The vector x^h in (3.1) is contravariant analytic, that is, x^h satisfies*

$$\mathcal{L}_x\, J_i^{\,h} = 0,$$

or, equivalently

$$\nabla_j x_i = J_j^{\,r} J_i^{\,s}\, \nabla_r x_s\,,$$

where $\mathcal{L}_x$ denotes the operator of Lie derivation with respect to x^h.

(III) *There exist a gradient vector* ρ_i *and a symmetric tensor* f_{jk} *such that for some functions* Φ *and* φ.

$$f_{r[j}\, J_{i]}{}^{r} = 0$$

and

$$\nabla_j \rho_i = -\rho_j \rho_i + \tilde{\rho}_j \tilde{\rho}_i + f_{ji} + \Phi g_{ji} + \varphi (\sigma_j \sigma_i + \tilde{\sigma}_j \tilde{\sigma}_i).$$

BIBLIOGRAPHY

[1] T. ADATI (1963) – *On subprojective spaces*, I, « Tôhoku Math. J. », *3*, 159–173.

[2] B. KAGAN (1930) – *Sur les espaces sous-projectifs*, « C.R. Acad. Sci. Paris », *191*, 548–550.

[3] M. MATSUMOTO (1969) – *On Kählerian spaces with parallel or vanishing Bochner curvature tensor*, « Tensor N.S. », *20*, 25–28.

[4] M. MATSUMOTO and S. TANNO (1973) – *Kählerian spaces with parallel or vanishing Bochner curvature tensor*, « Tensor N.S. », *27*, 291–294.

[5] S. TACHIBANA (1967) – *On the Bochner curvature tensor*, « Nat. Sci. Rep. Ochanomizu Univ. », *18*, 15–19

[6] S. YAMAGUCHI and T. ADATI – *On holomorphically subprojective Kählerian manifolds* I, to appear in « Ann. di Math. Pura ed Appl. ».

[7] K. YANO (1965) – *Differential geometry on complex and almost complex spaces*, Pergamon Press.

[8] K. YANO and S. BOCHNER (1953) - *Curvature and Betti numbers*, « Ann. of Math. Stud. », *32*.