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On a certain second order differential equation

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Equazioni differenziali ordinarie. — *On a certain second order differential equation.* Nota di ROLF REISSIG, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Questa Nota riguarda le soluzioni periodiche, con valore medio nullo, dalle equazioni di Liénard generalizzate.

Si rileva che una condizione posta da *Ezeilo* per la forza di richiamo è restrittiva perché il corrispondente termine deve essere allora lineare e si dimostra che il teorema di *Ezeilo* può essere dimostrato come applicazione di ben noti teoremi sul comportamento asintotico delle soluzioni dell'equazione di *Liénard*.

In a recent paper [1] Ezeilo considers the generalized Liénard equation

$$x'' + f(x)\,x' + g(x) = e(t) \equiv e(t + 2\pi) \tag{1}$$

where f, g, e are continuous functions on $\mathbf{R}$. Assuming that the mean value of the forcing term is equal zero he looks for a solution of class

$$\mathrm{H} = \left\{ \varphi(t) \in C^1(\mathbf{R}) \mid \varphi(t) \equiv \varphi(t + 2\pi),\ \int_0^{2\pi} \varphi(t)\,dt = 0 \right\}.$$

Ezeilo's assertion is as follows:

Equation (1) *admits at least one solution of class* H *when*

(i) $\displaystyle\int_0^{2\pi} e(t)\,dt = 0$

(ii) $\displaystyle\int_0^{2\pi} g(\varphi(t))\,dt = 0 \qquad \forall \varphi \in \mathrm{H}$

(iii) $f(x) \geqq a > 0 \qquad \forall x \in \mathbf{R}.$

Remark. Condition (iii) can be replaced by

$$f(x) \leqq a < 0 \qquad \forall x \in \mathbf{R}$$

since the transformation $t \to -t$ is allowed.

The purpose of the present Note is to derive from condition (ii) that the restoring term must be linear:

$$g(x) = bx \qquad (b \text{ a real constant}).$$

(*) Nella seduta del 10 aprile 1976.

However, Ezeilo's statement in the special case

$$x'' + f(x)\,x' + bx = e(t) \equiv e(t + 2\pi) \tag{2}$$

is an immediate consequence of some well-known results on the existence of periodic solutions provided that $b \neq 0$. At first, let us mention these results.

THEOREM 1 (Cp. [4], [6]).

All solutions of equation (2) *are bounded for* $t \geqq 0$ *when*

(i) $\int_0^{2\pi} e(t)\,\mathrm{d}t = 0$

(ii) $\lim_{|x| \to \infty} \operatorname{sgn} x \int_0^x f(u)\,\mathrm{d}u = +\infty$

(*which is fulfilled, for instance, in case* $f(x) \geqq a > 0$ $(|x| \geqq h)$)

(iii) $b \geqq 0$.

Note. The boundedness of solutions ensures the existence of a periodic solution (by virtue of a theorem of Massera [2]) provided that the initial value problem has a uniquely determined solution continuously depending upon the initial values.

THEOREM 2 (Cp. [3], [5]).

Equation (2) *where* $f(x)$ *is an arbitrary continuous function possesses at least one periodic solution when*

(i) $b < 1$

(ii) $\int_0^{2\pi} e(t)\,dt = 0$ *in case* $b = 0$.

Note. The proof of Theorem 2 given in [5] can be modified in a simple way in order to obtain Ezeilo's assertion in the exceptional case $g(x) \equiv 0$ (i.e. $b = 0$). Following [5] we choose an arbitrary constant $k \in (0, 1)$, and we consider the auxiliary equation

$$x'' + kx = \mu\,\{e(t) + kx - f(x)\,x'\}\,, \qquad 0 \leqq \mu < 1. \tag{3}$$

Let $G(t) \equiv G(t + 2\pi)$ be the (piecewise smooth) Green's function of the differential operator on the left which belongs to periodic boundary conditions; then the periodic solutions of (3) are the continuous solutions of the integral equation

(4) $$x(t) = \mu A\{x\} = \mu \int_0^{2\pi} G(t-s)\,[e(s) + kx(s)]\,ds - \mu \int_0^{2\pi} G'(t-s) \cdot F(x(s))\,ds \quad , \quad F(x) = \int_0^x f(u)\,du.$$

We calculate:

$$\int_0^{2\pi} A\{x\}(t)\,dt = \int_0^{2\pi} G(t)\,dt \int_0^{2\pi} [e(t) + kx(t)]\,dt.$$

Equation (4) is considered as an operator equation in the Banach space $X = \{\varphi(t) \in C^0(\mathbf{R}) \mid \varphi(t + 2\pi) \equiv \varphi(t)\}$ supplied with the supremum norm. In [5] the existence of a fixed point of A, $x(t) = A\{x\} \in X$ (which corresponds to a periodic solution of (2)), is proved by means of an a priori estimate of all solutions $x(t) = \mu A\{x\}$, $0 \leqq \mu < 1$. Of course, this estimate is valid, too, when X is replaced by its subspace X_0 which is characterized by the condition $\int_0^{2\pi} \varphi(t)\,dt = 0$. Evidently, $A\{X_0\} \subset X_0$.

Now, let us consider a continuous function $g(x)$ with the property

(5) $$\int_{-\pi}^{+\pi} g(\varphi(t))\,dt = 0 \qquad \forall \varphi \in H,$$

and let us show:

a) $g(x) = -g(-x)\ \forall x \in \mathbf{R}^+$ (i.e. $g(x)$ and odd function),

b) $\dfrac{g(x)}{x} = \text{constant}\ \forall x \in \mathbf{R}^+$ (i.e. $g(x) = bx$ on $\mathbf{R}$).

a) *Assumption*: There is a $\xi \in \mathbf{R}^+$, $|g(\xi) + g(-\xi)| = \rho > 0$.

Define a function $\psi(t) \in C^1\left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]$:

$\psi(t) = \psi(-t)$ if $t \leqq 0$

$\psi(t) = \xi > 0$ if $0 \leqq t \leqq \tau$

$\psi(t)$ monotone-decreasing, $\xi \geqq \psi(t) \geqq 0$, if $\tau \leqq t \leqq \tau + \sigma$ (1)

$\psi(t) = 0$ if $\tau + \sigma \leqq t \leqq \frac{\pi}{2}$.

(1) For example: $\frac{\xi}{2} + \frac{\xi}{2} \sin\left[\frac{\pi}{2} + \frac{\pi}{\sigma}(t - \tau)\right]$.

Here, τ and σ are positive values with $\tau + \sigma < \frac{\pi}{2}$; they will be chosen in an adequate manner.

Define an odd function $\varphi(t) \in H$:

$$\varphi(t) = \psi\left(t - \frac{\pi}{2}\right) \qquad \text{if} \quad 0 \leqq t \leqq \pi$$

$$\varphi(t) = -\psi\left(t + \frac{\pi}{2}\right) \qquad \text{if} \quad -\pi \leqq t \leqq 0 .$$

Using condition (5) we obtain

$$0 = \int_{-\pi}^{+\pi} g(\varphi(t))\,\mathrm{d}t = 2g(0)(\pi - 2(\tau + \sigma))$$

$$+ 2\int_{\tau}^{\tau+\sigma} [g(\psi(t)) + g(-\psi(t))]\,\mathrm{d}t$$

$$+ 2\tau[g(\xi) + g(-\xi)];$$

introducing $\eta = \max_{|x| \leqq \xi} |g(x)|$ we estimate

$$0 \geqq \tau\rho - \eta(\pi - 2\tau) > 0 \qquad \text{(contradiction!)}$$

if

$$\frac{\eta\pi}{2\eta + \rho} < \tau < \frac{\pi}{2}\left(0 < \sigma < \frac{\pi}{2} - \tau\right).$$

b) *Assumption*: $0 < \xi_1 < \xi_2$, $\dfrac{g(\xi_1)}{\xi_1} = b_1 \neq b_2 = \dfrac{g(\xi_2)}{\xi_2}$.

Using $\xi = \xi_1$ we construct $\psi(t) = \psi_1(t)$ as before, and we denote $\dfrac{\xi_1}{\xi_2} = q\,(< 1)$.

Furthermore, we define a function $\psi_2(t) \in C^1\left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]$:

$$\psi_2(t) = \psi_2(-t) \qquad \text{if} \quad t \leqq 0$$

$$\psi_2(t) = q^{-1}\psi_1(q^{-1}t) \qquad \text{if} \quad 0 \leqq t \leqq q(\tau + \sigma)$$

$$\psi_2(t) = 0 \qquad \text{if} \quad q(\tau + \sigma) \leqq t \leqq \frac{\pi}{2}.$$

Finally, we determine $\varphi(t) \in H$:

$$\varphi(t) = \psi_1\left(t - \frac{\pi}{2}\right) \qquad \text{if} \quad 0 \leqq t \leqq \pi$$

$$\varphi(t) = -\psi_2\left(t + \frac{\pi}{2}\right) \qquad \text{if} \quad -\pi \leqq t \leqq 0;$$

$$\int_{-\pi}^{+\pi} \varphi(t)\,\mathrm{d}t = 2\left[\int_0^{\pi/2} \psi_1(t)\,\mathrm{d}t - \int_0^{\pi/2} \psi_2(t)\,\mathrm{d}t\right] =$$

$$= 2\left[\int_0^{\tau+\sigma} \psi_1(t)\,\mathrm{d}t - q^{-1}\int_0^{q(\tau+\sigma)} \psi_1(q^{-1}t)\,\mathrm{d}t\right] = 0.$$

By virtue of condition (5) we have

$$0 = \int_{-\pi}^{+\pi} g(\varphi(t))\,dt = 2\left[\int_0^{\tau+\sigma} g(\psi_1(t))\,dt - \int_0^{q(\tau+\sigma)} g(\psi_2(t))\,dt\right]$$

$$= 2\left[b_1 \int_0^{\tau} \psi_1(t)\,dt + \int_{\tau}^{\tau+\sigma} g(\psi_1(t))\,dt\right]$$

$$- 2\left[b_2 \int_0^{q\tau} \psi_2(t)\,dt + \int_{q\tau}^{q(\tau+\sigma)} g(\psi_2(t))\,dt\right]$$

$$= 2(b_1 - b_2)\int_0^{\pi/2} \psi_1(t)\,dt$$

$$+ 2\int_{\tau}^{\tau+\sigma} [g(\psi_1(t)) - b_1\psi_1(t)]\,dt$$

$$- 2\int_{q\tau}^{q(\tau+\sigma)} [g(\psi_2(t)) - b_2\psi_2(t)]\,dt.$$

Choosing $\frac{\pi}{4} \leqq \tau < \frac{\pi}{2}$, $0 < \sigma < \frac{\pi}{2} - \tau$ and denoting $\max_{|x|\leqq\xi_2} |g(x)| = \eta_2$ we estimate

$$|b_1 - b_2|\,\xi_1\,\frac{\pi}{4} \leqq \sigma\,(2\,\eta_2 + |b_1|\,\xi_1 + |b_2|\,\xi_2)\,.$$

This result is a contradiction if we suppose

$$\sigma < \frac{\pi\xi_1\,|b_1 - b_2|}{8\,\eta_2 + 4\,|b_1|\,\xi_1 + 4\,|b_2|\,\xi_2}\,.$$

REFERENCES

[1] J. O. C. EZEILO (1974) – *Some new criteria for the existence of periodic solutions of a certain second order differential equation*, « Atti Acc. Naz. Lincei, Rend. Cl. Sci. fis. mat. nat. », *56*, 675–683.

[2] J. L. MASSERA (1950) – *The existence of periodic solutions of systems of differential equations*, « Duke Math. J. », *17*, 457–475.

[3] J. MAWHIN (1972) – *An extension of a theorem of A. C. Lazer on forced nonlinear oscillations*, « J. Math. Analysis Appl. », *40*, 20–29.

[4] S. MIZOHATA and M. YAMAGUTI (1951) – *On the existence of periodic solutions of the nonlinear differential equation* $x'' + f(x)\,x' + g(x) = e(t)$. « Mem. Coll. Sci. Univ. Kyoto, Ser. A, Math. », *27*, 109–113.

[5] R. REISSIG (1975) – *Extension of some results concerning the generalized Liénard equation*, « Ann. Mat. Pura Appl. » *104*, 269–281.

[6] R. REISSIG, G. SANSONE and R. CONTI (1963) – *Qualitative Theorie nichtlinearer Differentialgleichungen*. Edizioni Cremonese, Roma.