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Isomorphisms of rank two torsion-free modules over a Dedekind domain

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Algebra. — *Isomorphisms of rank two torsion-free modules over a Dedekind domain.* Nota di LUCIE DE MUNTER-KUYL, presentata (*) dal Corrisp. G. ZAPPA.

RIASSUNTO. — Scopo di questo lavoro è quello di stabilire condizioni necessarie e sufficienti affinché due moduli liberi da torsione sopra un dominio di Dedekind siano isomorfi. Queste condizioni sono applicate a varie classi di moduli e, in particolare, a moduli sopra un dominio principale.

1. INTRODUCTION

In this paper, we determine necessary and sufficient conditions for two torsion-free modules of rank two over a Dedekind domain to be isomorphic. Our main theorem is proved in Section 3 and is then applied to various classes of modules in the following sections. We extend several results of Beaumont-Pierce [1], Parr [5] and Richman [6].

2. DEFINITIONS AND NOTATIONS

Let A be a Dedekind domain, K its field of fractions, $\mathcal{P}$ the set of none zero prime ideals of A, $v_{\mathfrak{p}}$ the valuation of K associated with $\mathfrak{p} \in \mathcal{P}$, $K_{\mathfrak{p}}$ th- completion of K with respect to $v_{\mathfrak{p}}$, $A_{\mathfrak{p}}$ the local ring of A at $\mathfrak{p}$, u a uniformizing element of $A_{\mathfrak{p}}$, $\overline{A_{\mathfrak{p}}}$ the adherence of $A_{\mathfrak{p}}$ in $K_{\mathfrak{p}}$ and $\overline{U_{\mathfrak{p}}}$ the group of units of $\overline{A_{\mathfrak{p}}}$. Unless otherwise specified, we shall further adopt the terminology of Bourbaki [2].

We call *superdivisor of* A any mapping μ from $\mathcal{P}$ to $\overline{\mathbf{N}} = \mathbf{N} \cup \{\infty\}$; it is said to be *finite* if $\sum_{\mathfrak{p} \in \mathcal{P}} \mu(\mathfrak{p}) < \infty$. Multiplication is defined by $(\mu\mu')(\mathfrak{p}) = \mu(\mathfrak{p}) + \mu'(\mathfrak{p})$, with $\infty + n = \infty$ for all n. If there exist two finite superdivisors m and m' such that $m\,\mu = m'\,\mu'$, then μ and μ' are said to be of the same *type*. The type of μ is denoted $\tau(\mu)$. A type is *idempotent* if it is the type of some (unique) idempotent superdivisor. We say that $\tau \leq \tau'$ if there exist μ and μ' such that $\tau = \tau(\mu)$, $\tau' = \tau(\mu')$ and μ divides μ'. The GCD of μ and μ' is denoted $[\mu, \mu']$ and we set $[\tau, \tau'] = \tau([\mu, \mu'])$. The type consisting of all finite superdivisors is called *the finite type* and is denoted $\tau(1)$. We define zero by $0(\mathfrak{p}) = \infty$ for all $\mathfrak{p}$ and call $\tau(0)$ *the infinite type*.

If $\tau(\mu) = \tau(\mu')$ and if we set $\infty - \infty = 0$, then $\prod_{\mathfrak{p} \in \mathcal{P}} \mathfrak{p}^{\mu(\mathfrak{p}) - \mu'(\mathfrak{p})}$ is a fractional ideal of A. A finer equivalence relation is defined if we require

(*) Nella seduta del 10 aprile 1976.

that $\prod_{\mathfrak{p}\in\mathscr{P}} \mathfrak{p}^{\mu(\mathfrak{p})-\mu'(\mathfrak{p})}$ be a principal fractional ideal. The class of μ with respect to this relation is denoted $\mathscr{C}(\mu)$.

Let N be a A-submodule of K and let $x \in N$, $x \neq 0$. Let $h_{\mathfrak{p}}^{N}(x) = v_{\mathfrak{p}}(x) - v_{\mathfrak{p}}(N)$ and $h(N, x) : \mathfrak{p} \mapsto h_{\mathfrak{p}}^{N}(x)$. If $y \in N$ and $y \neq 0$, then $h(N, x)$ and $h(N, y)$ belong to the same class $\mathbf{C}(N)$ and, a fortiori, to the same type $\tau(N)$. Moreover, there is a bijective correspondence between the isomorphism classes of rank one A-modules and the classes of superdivisors of A. A module N is a fractional ideal if and only if $\tau(N) = \tau(1)$ and a principal fractional ideal if $\mathscr{C}(N) = \mathscr{C}(1)$, where $1(\mathfrak{p}) = 0$ for all $\mathfrak{p}$.

Let $\eta = (\eta(\mathfrak{p}))$ be a restricted adèle of A. If $\eta \in \prod_{\mathfrak{p}\in\mathscr{P}} \overline{A}_{\mathfrak{p}}$, we set $v_{\mathfrak{p}}(\eta) = v_{\mathfrak{p}}(\eta(\mathfrak{p}))$ and $v(\eta) : \mathfrak{p} \mapsto v_{\mathfrak{p}}(\eta)$. Let μ be a superdivisor. We say that two pairs (η_1, η_2) and (η_1', η_2') of elements of $\prod_{\mathfrak{p}\in\mathscr{P}} \overline{A}_{\mathfrak{p}}$ are *μ-equivalent* if (1) $v(\eta_i) = v(\eta_i')$, $i = 1, 2$; (2) $\mu v(\eta_1) v(\eta_2)$ divides $v(\eta_1 \eta_2' - \eta_2 \eta_1')$.

We denote by $\Omega = \Omega(\eta_1, \eta_2) = (\Omega(\mathfrak{p}))_{\mathfrak{p}\in\mathscr{P}}$ the μ-class of (η_1, η_2). The pair $(v(\eta_1), v(\eta_2))$ does not depend on the representative of Ω; we call it the *height of* Ω and denote it by $h(\Omega)$.

3. Isomorphism theorem

Let M be a rank two A-module. We identify M with the canonical image of $A \otimes M$ in $E_{\mathfrak{p}} = K_{\mathfrak{p}} \otimes M$ and denote by E, $M_{\mathfrak{p}}$ and $\overline{M}_{\mathfrak{p}}$ respectively the images of $K \otimes M$, $A_{\mathfrak{p}} \otimes M$ and $\overline{A}_{\mathfrak{p}} \otimes M$ in $E_{\mathfrak{p}}$.

Let $\langle x \rangle$ be the pure A-submodule of M generated by a non-zero element x, let $h_{\mathfrak{p}}^{M}(x) = \sup\{k \in \mathbf{N} ; u^{-k} x \in M_{\mathfrak{p}}\} = \sup\{k \in \mathbf{N} ; u^{-k} x \in A_{\mathfrak{p}} \langle x \rangle\}$ and let $h(M, x) : \mathfrak{p} \mapsto h_{\mathfrak{p}}^{M}(x)$.

Let x_1 and x_2 be independent elements of M and let $\mu_i = h(M, x_i)$, $i = 1, 2$. In [3], we have defined a complete system of invariants (μ, Ω) of the triple (M, x_1, x_2) where μ characterizes the structure of the rank one torsion A-module $M/(\langle x_1 \rangle + \langle x_2 \rangle)$ and where Ω is a μ-class whose height $h(\Omega) = (\mu_2, \mu_1)$. We denoted this by $\mathrm{inv}(M, x_1, x_2) = (\mu, \Omega)$. It was shown that $\mu(\mathfrak{p}) = 0$ when $(\mu_1, \mu_2)(\mathfrak{p}) = \infty$.

The $\overline{A}_{\mathfrak{p}}$-module $\overline{M}_{\mathfrak{p}}$ is decomposable into the direct sum of two rank one submodules. In [4], we have associated with the reference pair (x_1, x_2) some decompositions of $\overline{M}_{\mathfrak{p}}$, expressed in terms of μ and Ω, as well as some classes of matrices of $GL(2, K_{\mathfrak{p}})$. Particular representatives of these classes were also defined in [4] and will be used here.

Suppose M and M′ are two isomorphic rank two A-submodules of E. Let (x_1, x_2) (resp. (x_1', x_2')) be a reference pair in M (resp. M′); let $\mathrm{inv}(M, x_1, x_2) = (\mu, \Omega)$ (resp. $\mathrm{inv}(M', x_1', x_2') = (\mu', \Omega')$). If f is an isomorphism of M′ onto M and if $y_1 = f(x_1')$ and $y_2 = f(x_2')$, then $\mathrm{inv}(M, y_1, y_2) = (\mu', \Omega')$. We are thus led to investigate the way in which a change of reference pair affects the invariants of a triple.

For some $r_1, r_2, s_1, s_2 \in K$, we have $y_1 = r_1 x_1 + r_2 x_2$, $y_2 = s_1 x_1 + s_2 x_2$ and $B = \begin{pmatrix} r_1 & r_2 \\ s_1 & s_2 \end{pmatrix} \in GL(2, K)$. We now show that

(*a*) $\mu_1'(\mathfrak{p}) = \inf(v_\mathfrak{p}(r_1\eta_2 - r_2\eta_1), \mu(\mathfrak{p}) + v_\mathfrak{p}(r_1\eta_2), \mu(\mathfrak{p}) + v_\mathfrak{p}(r_2\eta_1))$,

(*b*) $\mu_2'(\mathfrak{p}) = \inf(v_\mathfrak{p}(s_1\eta_2 - s_2\eta_1), \mu(\mathfrak{p}) + v_\mathfrak{p}(s_1\eta_2), \mu(\mathfrak{p}) + v_\mathfrak{p}(s_2\eta_1))$,

(*c*) $(\mu'\mu_1'\mu_2')(\mathfrak{p}) = (\mu\mu_1\mu_2)(\mathfrak{p}) + v_\mathfrak{p}(\det B)$,

(*d*) $(\mu'\mu_1'\mu_2')(\mathfrak{p}) \leq v_\mathfrak{p}(\eta_1'(r_1\eta_2 - r_2\eta_1) + \eta_2'(s_1\eta_2 - s_2\eta_1))$,

for all $\mathfrak{p} \in \mathscr{P}$, $(\eta_1, \eta_2) \in \Omega$ and $(\eta_1', \eta_2') \in \Omega'$.

These relations are trivially satisfied if $\mu_1'(\mathfrak{p}) = \mu_2'(\mathfrak{p}) = \infty$.

Let $\mathfrak{p}$ be a fixed ideal and set $\eta_i(\mathfrak{p}) = \eta_i$, $i = 1, 2$. Let $\mu_1'(\mathfrak{p}) < \infty$ and $\mu_2'(\mathfrak{p}) = \infty$. Thus $\mu'(\mathfrak{p}) = 0$ and $(\mu\mu_1\mu_2)(\mathfrak{p}) = \infty$.

Suppose $\mu(\mathfrak{p}) = \infty$ and hence $(\mu_1\mu_2)(\mathfrak{p}) < \infty$. Let t_1 (resp. t_1') be an element of K such that $v_\mathfrak{p}(t_1) = -\mu_1(\mathfrak{p})$ (resp. $v_\mathfrak{p}(t_1') = -\mu_1'(\mathfrak{p})$) and $v_\mathfrak{q}(t_1)$, $v_\mathfrak{q}(t_1') \geq 0$, for all $\mathfrak{q} \in \mathscr{P}$, $\mathfrak{q} \neq \mathfrak{p}$. If $X_1 = \eta_1 x_1 + \eta_2 x_2$ (resp. $X_1' = x_2'$) and $Y_1 = t_1 x_1$ (resp. $Y_1' = t_1' x_1'$), then $K_\mathfrak{p} X_1 \oplus \overline{A}_\mathfrak{p} Y_1$ (resp. $K_\mathfrak{p} X_1' \oplus \overline{A}_\mathfrak{p} Y_1'$) is a decomposition of $\overline{M}_\mathfrak{p}$ associated with (x_1, x_2) (resp. (x_1', x_2')) (see [4]) to which corresponds the matrix

$$W_\mathfrak{p} = \begin{pmatrix} 0 & t_1^{-1} \\ \eta_2^{-1} & -\eta_1(\eta_2 t_1)^{-1} \end{pmatrix} \quad \left(\text{resp. } W_\mathfrak{p}' = \begin{pmatrix} 0 & t_1'^{-1} \\ 1 & 0 \end{pmatrix}\right)$$

giving the components of x_1 and x_2 (resp. x_1' and x_2') in the basis $\{X_1, Y_1\}$ (resp. $\{X_1', Y_1'\}$) of $E_\mathfrak{p}$. The matrix $W_\mathfrak{p}'^{-1} B W_\mathfrak{p}$ is of the form $\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} \in GL(2, K_\mathfrak{p})$, with $\delta \in \overline{U}_\mathfrak{p}$. Computation of its coefficients gives $v_\mathfrak{p}(r_1\eta_2 - r_2\eta_1) = \mu_1'(\mathfrak{p})$ and $v_\mathfrak{p}(s_1\eta_2 - s_2\eta_1) = \infty$.

Assuming that $\mu_2(\mathfrak{p}) = \infty$. and thus $\mu_1(\mathfrak{p}) < \infty$ and $\mu(\mathfrak{p}) = 0$, we would obtain $v_\mathfrak{p}(r_1\eta_2) = \mu_1'(\mathfrak{p})$ and $s_1 = 0$.

Relations (a) to (d) are then easily completed.

The case where $(\mu_1'\mu_2')(\mathfrak{p}) < \infty$ and $\mu'(\mathfrak{p}) = \infty$, and the case where $(\mu'\mu_1'\mu_2')(\mathfrak{p}) < \infty$ are treated similarly.

Conversely, let $B = \begin{pmatrix} r_1 & r_2 \\ s_1 & s_2 \end{pmatrix} \in GL(2, K)$ satisfy relations (a) to (d), and let $y_1 = r_1 x_1 + r_2 x_2$ and $y_2 = s_1 x_1 + s_2 x_2 \in E$. Let N be the (unique) A-submodule of rank two of E, containing y_1 and y_2, and such that $\text{inv}(N, y_1, y_2) = (\mu', \Omega')$ (see [3]). The module N is thus isomorphic to M'.

Denote by $\langle Z_1, Z_2 \rangle$ (resp. $\langle Z_1', Z_2' \rangle$) a decomposition of $\overline{M}_\mathfrak{p}$ (resp. $\overline{N}_\mathfrak{p}$) associated with (x_1, x_2) (resp. (y_1, y_2)), and set $Z_1' = \alpha Z_1 + \beta Z_2$ and $Z_2' = \gamma Z_1 + \delta Z_2$, where $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = C_\mathfrak{p} \in GL(2, K_\mathfrak{p})$. Then, for some $W_\mathfrak{p}$ (resp. $W_\mathfrak{p}'$) belonging to the class of matrices associated with (M, x_1, x_2) (resp. (N, y_1, y_2)), we have

$$\begin{pmatrix} Z_1' \\ Z_2' \end{pmatrix} = W_\mathfrak{p}'^{-1} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = W_\mathfrak{p}'^{-1} B \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = W_\mathfrak{p}'^{-1} B W_\mathfrak{p} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = C_\mathfrak{p} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}.$$

If $(\mu\mu_1\mu_2)(\mathfrak{p}) = \infty$, then it follows from (a), (b), (c), (d) that $C_\mathfrak{p}$ satisfies the conditions (stated in [4], [1]) which ensure that $\langle Z_1', Z_2' \rangle$ is a decomposition of $\overline{M}_\mathfrak{p}$, and thus $\overline{N}_\mathfrak{p} = \overline{M}_\mathfrak{p}$. If $(\mu\mu_1\mu_2)(\mathfrak{p}) < \infty$, then a simple computation which we shall omit shows that $C_\mathfrak{p}$ is again the matrix corresponding to a change of decomposition of $\overline{M}_\mathfrak{p}$. Therefore, $\overline{N}_\mathfrak{p} = \overline{M}_\mathfrak{p}$ for all $\mathfrak{p} \in \mathscr{P}$. Thus $N = M$ and we have proved

3.1. THEOREM. *Let* $\mathrm{inv}(M, x_1, x_2) = (\mu, \Omega)$ (resp. $\mathrm{inv}(M', x_1', x_2') = (\mu', \Omega')$) *and* $(\eta_1, \eta_2) \in \Omega$ (resp. $(\eta_1', \eta_2') \in \Omega'$). *Then* M *is isomorphic to* M' *if and only if there exists a matrix* $B = \begin{pmatrix} r_1 & r_2 \\ s_1 & s_2 \end{pmatrix} \in GL(2, K)$ *such that*

(a) $\mu_1' = [v(r_1\eta_2 - r_2\eta_1), \mu v(r_1\eta_2), \mu v(r_2\eta_1)]$,

(b) $\mu_2' = [v(s_1\eta_2 - s_2\eta_1), \mu v(s_1\eta_2), \mu v(s_2\eta_1)]$,

(c) $\mu'\mu_1'\mu_2' = \mu\mu_1\mu_2 v(\det B)$,

(d) $\mu'\mu_1'\mu_2'$ *divides* $v(\eta_1'(r_1\eta_2 - r_2\eta_1) + \eta_2'(s_1\eta_2 - s_2\eta_1))$ [1].

Let $x \in M$, $x \neq 0$. The type of the superdivisor $h(M, x)$ is also the type of the rank one A-module $\langle x \rangle$. Applying Theorem 3.1, we can now determine both the type and the class of any rank one A-submodule of M. Indeed, we have.

3.2. PROPOSITION. *Let* $k \in K^*$ *and let* $N(k) = K(x_1 + kx_2) \cap M$. *Let* $\sigma_k : \mathfrak{p} \mapsto \inf(\mu(\mathfrak{p}) + [\mu_1, v(k)\mu_2](\mathfrak{p}), v_\mathfrak{p}(\eta_2 - k\eta_1))$, *Then* $\mathscr{C}(N(k)) = \mathscr{C}(\sigma_k)$.

As a pure submodule, $N(k)$ is generated by any $x = ax_1 + bx_2$, with $a, b \in A$ and $ba^{-1} = k$. The proposition then follows from Theorem 3.1 (a).

The type of $N(k)$ can be expressed more simply as follows:

3.3. COROLLARY. *Let* $\lambda_k : \mathfrak{p} \mapsto \inf((\mu\mu_0)(\mathfrak{p}), v_\mathfrak{p}(\eta_2 - k\eta_1))$, *where* $\mu_0 = [\mu_1, \mu_2]$. *Then* $\tau(N(k)) = \tau(\lambda_k)$.

The set of all types $\tau(N)$, where N runs through all rank one pure submodules of M is called the *type set* of M and will be denoted $T(M)$.

4. TYPE, COTYPE AND EMBEDDING RATIO

From Theorem 3.1 (c), it follows immediately that the class $\mathscr{C}(\mu\mu_1\mu_2)$ is invariant under a change of reference pair. And therefore, so is the type $\tau(\mu\mu_1\mu_2)$.

Let $\mathscr{P}_0 = \{\mathfrak{p} \in \mathscr{P}; \mu_1(\mathfrak{p}) = \mu_2(\mathfrak{p}) = \infty\}$, $\mathscr{P}_1 = \{\mathfrak{p} \in \mathscr{P} - \mathscr{P}_0; (\mu\mu_1\mu_2)(\mathfrak{p}) = \infty\}$ and $\mathscr{P}_2 = \{\mathfrak{p} \in \mathscr{P}; (\mu\mu_1\mu_2)(\mathfrak{p}) < \infty\}$. Then Theorem 3.1 shows that

(1) Theorem 3.1 was proved in the author's doctoral dissertation supervised by Prof. J. Tits and presented at the University of Brussels in January 1971.

the non-empty sets among $\mathscr{P}_0, \mathscr{P}_1, \mathscr{P}_2$ form a partition of $\mathscr{P}$ which is also independent of the choice of x_1 and x_2. If $\mathfrak{p} \in \mathscr{P}_0$, then $\mu(\mathfrak{p}) = 0$ and $\Omega(\mathfrak{p})$ is reduced to the pair $(0, 0)$. If $S = A - \bigcup_{\mathfrak{p} \notin \mathscr{P}_0} \mathfrak{p}$, then M is a S^{-1} A-module having at every prime ideal $S^{-1}\mathfrak{p}$, with $\mathfrak{p} \in \mathscr{P}_1 \cup \mathscr{P}_2$, the same invariants as the A-module M, and for which the corresponding $\mathscr{P}_0$ is empty. From now on, we shall suppose that $\mathscr{P}_0 = \emptyset$, i.e. $\mu_0(\mathfrak{p}) < \infty$, for all $\mathfrak{p}$.

From Corollary 3.3, we deduce easily that $\tau_0 = \tau(\mu_0)$ is independent of the choice of (x_1, x_2) and that the types of rank one pure submodules of M which are distinct from τ_0 are pairwise incomparable, each of them being represented by only one pure submodule. Obviously, the class of μ_0 depends in general on the choice of x_1 and x_2.

When T(M) is a finite set, then $\tau_0 \in T(M)$. In the case of a decomposable module, we have $M = \langle x_1 \rangle \oplus \langle x_2 \rangle$ for some pair (x_1, x_2) and thus $\mu = 1$. Then $\tau(\lambda_k) = \tau_0$, for all $k \in K^*$, and hence card $T(M) \leq 3$. In general, it results from Corollary 3.3 that $\tau(\lambda_k) \neq \tau_0$ if and only if at least one of the following conditions is satisfied:

4.1. There exists $\mathfrak{p} \in \mathscr{P}_1$ such that $\eta_2(\mathfrak{p}) = k\eta_1(\mathfrak{p})$.

4.2. For infinitely $\mathfrak{p} \in \mathscr{P}$, we have $\mu(\mathfrak{p}) > 0$ and $v_{\mathfrak{p}}(\eta_2 - k\eta_1) > \mu_0(\mathfrak{p})$.

Following Parr [5], we call τ_0 the *type of* M, while we shall call $\tau(\mu\mu_1\mu_2)$ the *cotype of* M (Parr's cotype actually corresponds to $\tau(\mu\mu_1\mu_2\mu_0^{-1})$). Let $\rho(\mathfrak{p}) = 0$ if $(\mu_1\mu_2)(\mathfrak{p}) = \infty$ and $\rho(\mathfrak{p}) = \eta_2(\mathfrak{p})(\eta_1(\mathfrak{p}))^{-1}$ if $(\mu_1\mu_2)(\mathfrak{p}) < \infty$. Consider the A-module $R = \prod_{\mathfrak{p} \in \mathscr{P}} R_{\mathfrak{p}}$, where $R_{\mathfrak{p}} = 0$ if $(\mu_1\mu_2)(\mathfrak{p}) = \infty$ and $R_{\mathfrak{p}} = (\mathfrak{p}\overline{A}_{\mathfrak{p}})^{\mu_1(\mathfrak{p}) - \mu_2(\mathfrak{p})} / (\mathfrak{p}\overline{A}_{\mathfrak{p}})^{\mu_1(\mathfrak{p}) - \mu_2(\mathfrak{p}) + \mu(\mathfrak{p})}$ if $(\mu_1\mu_2)(\mathfrak{p}) < \infty$. Let $\overline{\rho} = (\overline{\rho(\mathfrak{p})})_{\mathfrak{p} \in \mathscr{P}}$, where $\overline{\rho(\mathfrak{p})}$ denotes the canonical image of $\rho(\mathfrak{p})$ in $R_{\mathfrak{p}}$. We proved in [3] that μ and Ω can be replaced by μ_1, μ_2, μ and $\overline{\rho}$, and it will be clear in Section 8 that, in the case of a torsion-free abelian group and for a suitable choice of x_2, Parr's *embedding ratio* $(x_1 : x_2)$ of x_1 to x_2 is precisely $-\overline{\rho}$.

5. MODULES WITH AT LEAST TWO INCOMPARABLE TYPES

Let M and M' be rank two submodules of E with the same type set T. Suppose card $T \geq 3$ and let $\tau_1, \tau_2 \in T$ be incomparable types. Let $x_i \in M$ and $x_i' \in M'$ be such that $\tau(\mu_i') = \tau(\mu_i') = \tau_i$, $i = 1, 2$. If it exists, an isomorphism f of M' onto M maps $\langle x_i' \rangle$ onto $\langle x_i \rangle$ and hence $\mathscr{C}(\mu_i) = \mathscr{C}(\mu_i')$; therefore, x_1' and x_2' can be chosen such that $\mu_i' = \mu_i$. In addition, we must have $\mu' = \mu$, since μ is independent of the choice of the non-zero elements x_1 and x_2 of $\langle x_1 \rangle$ and $\langle x_2 \rangle$ respectively. Let $f(x_1') = rx_1$ and $f(x_2') = sx_2$, where $r, s \in K^*$. Applying Theorem 3.1, we obtain $(s\eta_1, r\eta_2) \in \Omega'$ for any $(\eta_1, \eta_2) \in \Omega$ and $\overline{\rho'} = \overline{rs^{-1}\rho}$.

Now, let two incomparable types τ_1 and τ_2 be given and let μ_1 and μ_2 be superdivisors of types τ_1 and τ_2 respectively. Let x_1 and x_2 be two fixed independent elements of E. Let $\mathcal{M}$ be the class of all A-submodules M of E containing x_1 and x_2, and such that $h(\mathrm{M}, x_i) = \mu_i$, $i = 1, 2$. An element of $\mathcal{M}$ is completely determined by its invariants μ and $\overline{\rho}$ relative to the reference pair (x_1, x_2) and any A-module of rank two containing elements of classes $\mathscr{C}(\mu_1)$ and $\mathscr{C}(\mu_2)$ is isomorphic to some module of $\mathcal{M}$.

We then have immediately

5.1. THEOREM. *Two modules* M *and* M′ *of* $\mathcal{M}$, *with invariants respectively* $(\mu, \overline{\rho})$ *and* $(\mu', \overline{\rho'})$, *are isomorphic if and only if*

(a) $\mu = \mu'$ *and*

(b) *there exist* $r, s \in \mathrm{K}^*$ *such that* $\overline{\rho'} = \overline{rs^{-1}\rho}$, *with* $v_{\mathfrak{p}}(r) = 0$ *if* $\mu_1(\mathfrak{p}) < \infty$ *and* $v_{\mathfrak{p}}(s) = 0$ *if* $\mu_2(\mathfrak{p}) < \infty$.

5.2. *Remark.* The module $\bigcap_{\mathrm{M} \in \mathcal{M}} \mathrm{M}$ is the only decomposable module in $\mathcal{M}$. Its invariants are $\mu = 1$ and $\overline{\rho} = 0$.

6. Decomposable modules

Let M be decomposable. If card T(M) = 3, it follows from Section 4 that $\langle x_1 \rangle$ and $\langle x_2 \rangle$ are uniquely determined by the condition that $\mu = 1$ with respect to the pair (x_1, x_2), and M is then the direct sum of two rank one pure submodules of incomparable types. When card T(M) ≤ 2, we have the following.

6.1. PROPOSITION. *Suppose* card T(M) ≤ 2. *Then* M *is decomposable if and only if there exists a reference pair* (x_1, x_2) *relative to which* $\tau(\mu) = \tau(1)$.

Necessity is obvious. We prove sufficiency. We can suppose $\tau_0 = \tau(\langle x_1 \rangle) \leq \tau(\langle x_2 \rangle)$. After possibly multiplying x_2 by an element of A, we shall even suppose $\mu_1(\mathfrak{p}) \leq \mu_2(\mathfrak{p})$, for all $\mathfrak{p}$. Let $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_k\}$ be the set of prime ideals for which $\mu(\mathfrak{p}) > 0$. As it was proved in [3], Lemma 2, M is generated by the set $\{x_1, x_2, t_1(\mathfrak{p}, m)\, x_1, t_2(\mathfrak{p}, n)\, x_2, s_i(a_1(i)\, x_1 + a_2(i)\, x_2)$, for all $\mathfrak{p} \in \mathscr{P}$ and all $m, n \in \mathbf{N}$ such that $0 < m \leq \mu_1(\mathfrak{p})$, $0 < n \leq \mu_2(\mathfrak{p})$, and where $i = 1, \cdots, k\}$, with $v_{\mathfrak{p}}(t_1(\mathfrak{p}, m)) = -m$, $v_{\mathfrak{p}}(t_2(\mathfrak{p}, n)) = -n$, $v_{\mathfrak{q}}(t_1(\mathfrak{p}, m))$, $v_{\mathfrak{q}}(t_2(\mathfrak{p}, n)) \geq 0$ if $\mathfrak{q} \neq \mathfrak{p}$, $v_{\mathfrak{p}_i}(s_i) = -(\mu\mu_1\mu_2)(\mathfrak{p}_i)$, $v_{\mathfrak{p}}(s_i) \geq 0$ if $\mathfrak{p} \neq \mathfrak{p}_i$, $v_{\mathfrak{p}_i}(a_1(i)) = \mu_2(\mathfrak{p}_i)$ and $v_{\mathfrak{p}_i}(a_2(i)) = \mu_1(\mathfrak{p}_i)$, and where $a_1(i)$ and $a_2(i)$ belong to A.

For each i, choose $b_i \in \mathrm{A}$ such that $v_{\mathfrak{p}_i}(1 - b_i s_i a_1(i)) = \mu(\mathfrak{p}_i)$ and therefore $v_{\mathfrak{p}_i}(b_i) = \mu(\mathfrak{p}_i) + \mu_1(\mathfrak{p}_i)$ and $v_{\mathfrak{p}_j}(b_i s_i a_2(i)\, a_1(j)) \geq \mu(\mathfrak{p}_i) + \mu_1(\mathfrak{p}_i)$ for all $j = 1, \cdots, k$, $j \neq i$. Let $r = \sum_{i=1}^{k} b_i s_i a_2(i)$ and $z = x_1 + r x_2$. Then $z = (1 - \sum_{i=1}^{k} b_i s_i a_1(i))\, x_1 + \sum_{i=1}^{k} b_i s_i (a_1(i)\, x_1 + a_2(i)\, x_2)$, where $\sum_{i=1}^{k} b_i s_i a_1(i) \in \mathrm{A}$, and thus $z \in \mathrm{M}$. Applying Theorem 3.1, one verifies easily that $\mathrm{M} = \langle z \rangle \oplus \langle x_2 \rangle$.

If card $T(M) = 2$, then $\langle x_2 \rangle$ is uniquely determined by the condition that its type strictly dominate τ_0.

Now, suppose M is finitely generated. Let (x_1', x_2') be a pair for which $\mu' = 1$. Let $r_1, r_2 \in K$ such that $v_\mathfrak{p}(r_1) \geq -\mu_1'(\mathfrak{p})$ for all $\mathfrak{p} \in \mathcal{P}$, $v_\mathfrak{p}(r_2) = -\mu_2'(\mathfrak{p})$ for the prime ideals $\mathfrak{p}$ (in finite number) such that $v_\mathfrak{p}(r_1) > -\mu_1'(\mathfrak{p})$, and $v_\mathfrak{p}(r_2) \geq 0$ elsewhere. Let $x_1 = r_1 x_1' + r_2 x_2'$. Then $\mu_1 = h(M, x_1) = 1$. As in the proof of Proposition 6.1, we can still choose $x_2 = s x_1 + x_2'$ such that $\mu = 1$ with respect to the pair (x_1, x_2) and therefore $\mathscr{C}(\mu_2) = \mathscr{C}(\mu\mu_1\mu_2)$, and we obtain the classical structure theorem

6.2. COROLLARY. *If* M *is finitely generated, then there exists a pair* (x_1, x_2) *such that* $M = Ax_1 \oplus \langle x_2 \rangle$, *where* $\langle x_2 \rangle$ *is isomorphic to an ideal of* A. *The class of* $\langle x_2 \rangle$ *is an invariant of* M.

7. MODULES WITH INFINITE COTYPE

Let M be a module with infinite cotype $\tau(\mu\mu_1\mu_2) = \tau(0)$. Then $\mathcal{P}_1 = \mathcal{P}$ and, from Theorem 3.1, we obtain $\mu_1'(\mathfrak{p}) = v_\mathfrak{p}(r_1\eta_2 - r_2\eta_1)$, $\mu_2'(\mathfrak{p}) = v_\mathfrak{p}(s_1\eta_2 - s_2\eta_1)$ and $\eta_1'(\mathfrak{p})(r_1\eta_2(\mathfrak{p}) - r_2\eta_1(\mathfrak{p})) + \eta_2'(\mathfrak{p})(s_1\eta_2(\mathfrak{p}) - s_2\eta_1(\mathfrak{p})) = 0$ for all $\mathfrak{p} \in \mathcal{P}$. This implies, in particular, that if M is decomposable, the $\rho(\mathfrak{p})$ belongs to K for all $\mathfrak{p} \in \mathcal{P}$ and takes at most two distinct values in K. The converse is easily seen to hold provided we assume beforehand that card $T(M) \leq 3$ (Otherwise, M could be an indecomposable module with 4 types). We thus have

7.1. PROPOSITION. *Let* M *be a module with infinite cotype and let* card $T(M) \leq 3$. *Then* M *is decomposable if and only if* $\rho(\mathfrak{p})$ *belongs to* K *for all* $\mathfrak{p} \in \mathcal{P}$ *and takes at most two distinct values in* K.

7.2. *Remark.* Let $\hat{K}_\mathfrak{p} = K_\mathfrak{p} \cup \{w_\mathfrak{p}\}$, where $w_\mathfrak{p}$ stands for $\alpha/0$, for any $\alpha \in K_\mathfrak{p}^*$, and $\alpha + w_\mathfrak{p} = w_\mathfrak{p}$, for all $\alpha \in \hat{K}_\mathfrak{p}$. It is then readily checked that a change B of reference pair induces, for each $\mathfrak{p} \in \mathcal{P}$, a homographic transformation $h_{\hat{B}}$ of $\hat{K}_\mathfrak{p}$, with $\hat{B} = \begin{pmatrix} -r_1 & r_2 \\ s_1 & -s_2 \end{pmatrix}$, and such that if we define $\hat{\rho}(\mathfrak{p}) = \rho(\mathfrak{p})$ when $\mu_2(\mathfrak{p}) < \infty$ and $\hat{\rho}(\mathfrak{p}) = w_\mathfrak{p}$ when $\mu_2(\mathfrak{p}) = \infty$, then $h_{\hat{B}}(\hat{\rho}(\mathfrak{p})) = \hat{\rho}'(\mathfrak{p})$.

8. MODULES OVER A PRINCIPAL DOMAIN

Suppose A is a principal domain. Then, for modules with at least two incomparable types, Theorem 5.1 can be given a simpler form:

8.1. THEOREM. *Let* M *and* M' *be indecomposable elements of* $\mathcal{M}$ *such that* $\mu = \mu'$. *Then* $M \simeq M'$ *if and only if there exists* $k \in K^*$ *such that* $v_\mathfrak{p}(k) = 0$ *when* $(\mu_1\mu_2)(\mathfrak{p}) < \infty$, *and* $\overline{\rho'} = \overline{k\rho}$. *In case* $\tau(\mu) > \tau(1)$, *if* k *exists, it is unique.*

Indeed, there exist $r, s \in K^*$ satisfying Theorem 5.1 (b) and such that $k = rs^{-1}$. If $\overline{k\rho} = \overline{k'\rho}$, then $v_{\mathfrak{p}}(k - k') \geq \mu(\mathfrak{p})$ whenever $\mu(\mathfrak{p}) > 0$, which implies $k = k'$ if $\tau(\mu) > \tau(1)$.

The case of modules with at most two types remains more difficult to deal with. However, some simplification will result from the fact that Parr's Proposition 2.1 [5] generalizes without change to give

8.2. PROPOSITION. *Let* N *be a* A*-submodule of* K *such that* $\tau(N) = \tau_0$, *where* τ_0 *is the type of some rank two* A*-module* M. *Then*

(a) *The* A*-modules* M *and* $N \otimes_A \mathrm{Hom}_A(N, M)$ *are isomorphic.*

(b) *The* A*-module* $\mathrm{Hom}_A(N, M)$ *is of rank two and has idempotent* type.

We may therefore confine ourselves to modules with idempotent type. As $\mathscr{P}_0$ is empty, this implies $\tau_0 = \tau(1)$. Then, there exists a pair (x_1, x_2) for which $\mu_0 = 1$. Furthermore, if $T(M) = \{\tau_0, \tau_1\}, \tau_0 < \tau_1$, then (x_1, x_2) can be chosen such that $\tau(h(M, x_1)) = \tau_1$ and $\mu_2 = 1$. When $T(M) = \{\tau_0\}$, (x_1, x_2) will be chosen such that $\mu_1 = \mu_2 = 1$. Following Parr, we define M to be *strongly reduced* if it contains no element of type equal to the cotype of M. If card $T(M) \geq 3$, then M is always strongly reduced. If card $T(M) \leq 2$, then M is strongly reduced if and only if it is indecomposable. The only strongly reduced modules which are decomposable are thus the direct sums of two rank one modules of incomparable types. If we apply Theorem 3.1 to these particular conditions, we obtain immediately Parr's Corollary 6.3, while Parr's Section 7 results, as a particular case, from our Section 7.

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