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On some function-geometric aspects of holomorph convex spaces

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RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

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Presiede il Presidente della Classe BENIAMINO SEGRE

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *On some function-geometric aspects of holomorph-convex spaces.* Nota di VO VAN TAN [(*)], presentata [(**)] dal Corrisp. G. ZAPPA.

RIASSUNTO. — In questa Nota si studia la distribuzione dei sottoinsiemi analitici compatti, di dimensione > 0, di uno spazio olomorficamente convesso X mediante la conoscenza dell'anello delle sue funzioni olomorfe globali. Le dimostrazioni complete compariranno altrove.

Unless the contrary is explicitly stated, all **C**-analytic spaces are assumed to be paracompact, non compact, reduced, irreducible and of **C**-dim $= n \geq 1$.

§ 1. THE VARIOUS TYPES OF HOLOMORPH-CONVEX SPACES

DEFINITION 1. *Let* $\pi : X \to Y$ *be a holomorphic map between* **C**-*analytic spaces and let*

$$\rho_x(\pi) := \dim_x X - \dim_x \pi^{-1}\pi(x) \qquad \textit{for} \quad x \in X.$$

Then the rank of π *denoted by* $RK\pi$ *is defined by*

$$RK\pi := \sup_{x \in X} \rho_x(\pi).$$

Certainly, $0 \leq RK\,\pi \leq \min(\dim X, \dim Y)$.

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Now, for $Y = \mathbf{C}^k$, $k \geq 1$, we have a second notion of rank, namely

DEFINITION 2. *Let* $f : X \longrightarrow \mathbf{C}^k$ *be a holomorphic map. Let* X' *be the open set of regular points of* X. *Then the rank of* f, *denoted by* $rk \cdot f$, *is defined to be the supremum of the rank of the jacobian matrix of* f *at all points of* X'.

Actually, these 2 notions coincide, namely

PROPOSITION 1 [3]. *Let* $f : X \longrightarrow \mathbf{C}^k$ *be a holomorphic map, then*

$$RKf = rk \cdot f.$$

DEFINITION 3. i) *Let* $f_i \in \Gamma(X, O_X)$ $1 \leq i \leq k$. *The* f_i *are said to be* analytically independent *if the Map*

$$f := (f_1, \cdots, f_k) : X \longrightarrow \mathbf{C}^k \quad \text{has maximal rank, i.e. } rk \cdot f = k.$$

ii) *The maximal number of analytically independent holomorphic functions on a* **C**-*analytic space* X (*denoted for short by* Mani (X)) *is the greatest integer* p *such that there exist* p *global holomorphic functions* $f_1, \cdots, f_p$ *on* X *which are analytically independent.*

DEFINITION 4. *A* **C**-*analytic space* X *is said to be holomorphically convex if for any closed discrete sequence* $\{x_n\}$, *there exists a global holomorphic function* f *on* X *such that*

$$\lim_{n \to \infty} |f(x_n)| = \infty ,$$

From Definition 3, it is clear that

i) *If* X *is Stein than* Mani (X) = dim X.

ii) *If* X *is compact then* Mani (X) = 0.

Our investigation will be on holomorph-convex spaces between Stein and compact spaces. First of all, let us mention the following important result, see [4] and [1].

THEOREM I (Remmert, Cartan).

Let X *be a holomorph-convex space, with* dim $X = n$, *then there exist*

i) *a Stein space* Y;

ii) *a proper, surjective and holomorphic map* $\pi : X \longrightarrow Y$ *such that the fibres of* π *are connected.*

iii) *and the induced mapping* $\pi_* : \Gamma(Y, O_Y) \longrightarrow \Gamma(X, O_X)$ *is an isomorphism. From now on the pair* (π, Y) *will be called the Remmert-Stein reduction of* X.

Remarks. i) It follows from Theorem I that $\dim Y \leq \dim X = n$. One of our next goals is to determine dim Y from the function geometric data on X.

ii) In Theorem I, one noticed that, for an open set $U \subset Y$, so that $\pi \mid V$, with $V := \pi^{-1}(U)$, is 1 to 1, then $\pi \mid V$ is biholomorph.

Definition 5. – *Let* X *be a holomorph-convex space*, $\dim X = n$, *with its Remmert-Stein reduction* (π, Y), *then*

i) X *is said to be of type* I, *if* $\dim Y = n$;

ii) *Otherwise* X *is said to be of type* II.

Examples. i) All Stein spaces, 1-convex spaces and more generally the proper modifications of Stein spaces (see Definitions 6 and 7 below) are of type I.

ii) Let $X = \mathbf{C}^n \times \mathbf{P}_m$ with $m \geq 1$, then X is of type II. From definitions 4 and 5, it is clear that

If X is of type I then $Mani(X) = n$

If X is of type II then $Mani(X) < n$.

Our next step is to provide the converses for these two facts.

§ 2. Classifications

The previous definitions, Proposition 1 and Theorem I give us

Lemma 1. *Let* X *be a holomorph-convex space with* $\mathrm{Mani}(X) = q \leq n$ (i.e. *there exists a map* $f: X \longrightarrow \mathbf{C}^q$ *with maximal rank*), *then there exist a Stein space Y and a proper surjective and holomorphic map* $\pi: X \longrightarrow Y$ *such that* $\mathrm{RK}\pi = rk \cdot f = q$

$$\begin{array}{ccc} X & \longrightarrow & \mathbf{C}^q \\ \downarrow & & \\ Y & & \end{array}$$

Moreover, for each $z \in X$, *the level set*

$$L_f(z) := \{x \in X \mid f(x) = f(z) \quad \textit{for all} \quad f \in \Gamma(X, O_X)\}$$

is an analytic subvariety in X *and all its connected components are compact.*

From Lemma 1 we easily obtain the following

Theorem I. *Let* X *be a holomorph-convex space with its Remmert-Stein reduction* (π, Y), *then*

$$\dim Y = q \quad \textit{iff} \quad \mathrm{Mani}(X) = q .$$

COROLLARY 1. *Let* X *be a holomorph-convex space, then*

X *is of type* I *iff* Mani (X) $= n$

X *is of type* II *iff* Mani (X) $< n$.

Before getting a clear picture of the distribution of compact analytic subvarieties in a given holomorph-convex space, we need the following two basic results, see [5].

THEOREM II (Remmert). *Let* $\pi : X \longrightarrow Y$ *be a holomorphic map between analytic spaces, then the set*

$$R(X) := \{x \in X \mid \rho_x(\pi) < RK\pi\}$$

is an analytic subvariety in X *of* codim ≥ 1.

Moreover if π is proper and surjective then $T(X) := \pi(R(X))$ is an analytic subvariety in Y, with $\dim T(X) \leq RK\pi - 2$.

THEOREM III (Remmert). *Let* X *be a holomorph-convex space, the set* S (X) : = {*union of all compact subvarieties of positive* dim *in* X} *is an analytic subvariety in* X.

From now on, we will call S (X) the *degeneracy variety* of X, R (X) the *rank subvariety* of X and T (X) the *critical subvariety* of Y. The next result will tell us how the S (X) and R (X) are related when X is holomorphically convex, namely we have

PROPOSITION 2. *Let* X *be a holomorph-convex space with its Remmert-Stein reduction* (π, Y), *then with respect to the map* π

$$R(X) \subseteq S(X) \subseteq X.$$

Examples. i) If X is Stein then $Y = X$ and $R(X) = S(X) = \emptyset$;

ii) Let X be the blowing up of $\mathbf{C}^n$ at the origin then $Y = \mathbf{C}^n$ and $R(X) = S(X) = \mathbf{P}_{n-1}$;

iii) If $X = \mathbf{C}^n \times \mathbf{P}_m$, $m \geq 1$ then $Y = \mathbf{C}^n$, $R(X) = \emptyset$ and $S(X) = X$.

Now our Theorem 1 will be strengthened with the following

THEOREM 2. *Let* X *be a holomorph-convex space with* Mani $(X) = q \leq n$ *and let* (π, Y) *be its Remmert-Stein reduction, then*

$$\dim \pi^{-1}(x) \begin{cases} = n - q & \text{if } x \in Y \setminus T(X) \\ > n - q & \text{if } x \in T(X) \end{cases}$$

where T (X) *is the critical subvariety in* Y.

COROLLARY 2. *Let* X *be a holomorph-convex space, then the following three conditions are equivalent.*

i) X *is of type* II;

ii) $R(X) \neq S(X)$;

iii) $S(X) = X$.

Remark. In general, neither R (X) nor S (X) are compact. A purely sheaf-theoretic approach was taken up in [6] in order to provide a necessary and sufficient condition for R (X) to be compact. Before giving a precise description for holomorph-convex spaces of type I, we need few more definitions.

DEFINITION 6. X *is called a proper modification of* Y, *if there exist* i) *a proper surjective and holomorphic map* $\pi : X \longrightarrow Y$;

ii) *proper analytic subvarieties* $S \subset X$ *and* $T \subset Y$ *with* $\dim S > \dim T$;

iii) π *induces by restriction, an isomorphism* $X \setminus S \xrightarrow{\sim} Y \setminus T$.

Example. Let $\mathbf{C}^m \subset \mathbf{C}^n$, with $m \leq n - 2$ and let X be the blowing up of $\mathbf{C}^n$ along $\mathbf{C}^m$, then X is a proper modification of $Y = \mathbf{C}^n$ with $T = \mathbf{C}^m$ and $S = \mathbf{C}^m \times \mathbf{P}_q$ where $q = n - m - 1$.

DEFINITION 7. *A* **C**-*analytic space* X *is said to be* 1-*convex if* X *is a proper modification of a Stein space* Y *at finitely many points* (*i.e. in the terminology of Definition* 6, T *consists of finitely many points*).

The example ii) after Proposition 2 provides an example of 1-convex space. There T = origin.

From remark ii) after Theorem I, Theorem 1 and Theorem III, it follows readily

THEOREM 3. *Let* X *be a holomorph-convex, then the following conditions are equivalent*:

i) X *is of type* I;

ii) X *is a proper modification of a Stein space*;

iii) R (X) = S (X);

iv) S (X) $\neq$ X.

Remark. The equivalence of i) and ii) has been pointed out in [2] and [4].

DEFINITION 8. *Let* X *be a holomorph-convex space with its Remmert-Stein reduction* (π, Y) *then* X *is called* non-degenerate *if* π (S (X)) *is a discrete set in* Y, *with* S (X) *the degeneracy variety in* X.

Exemple. Let E be a discrete closed set in $\mathbf{C}^n$, with $n \geq 2$ and let X be the blowing up of $\mathbf{C}^n$ along E then X is a non degenerate holomorph-convex manifold.

Remark. Certainly if X is non degenerate, X is of type I. The converse is not true, see the example after Definition 6. Meanwhile, from Definition 8, for a given holomorph-convex space X of type I, one has

i) X is Stein iff S (X) = $\emptyset$

ii) X is 1-convex iff S (X) is compact

iii) X is non degenerate iff all the connected components of S (X) are compact.

The view point of [6] was to look for a necessary and sufficient condition for the compactness of certain subvarieties of S (X), namely the ones which contained all compact subvarieties of dim $\geq q > 1$ in X. The following result tells us that in effect, the non degenerate holomorph-convex spaces are not far away from 1-convex spaces.

PROPOSITION 3 (see also [2]). *Let* X *be a holomorph-convex space* X *is non degenerate iff* $X = \bigcup_{i=1} X_i$ *with* X_i *with* $X_i \Subset X_{i+1}$ *and the* X_i *are* 1*-convex.*

§ 3. LOWER DIMENSIONAL CASES

Let's round off this discussion by giving a complete classification for 1-dimensional analytic spaces and 2-dimensional holomorph–convex spaces.

PROPOSITION 4. *Let* X *be a* 1-dim **C**-*analytic space.*

i) X *is Stein iff* Mani (X) = 1.

ii) X *is compact iff* Mani (X) = 0.

Remark. Proposition 4 is not true if we don't assume X to be irreducible. Clearly, there exist reducible 1-dimensional **C**-analytic spaces which are neither compact nor Stein.

From Theorems II and 3, we obtain easily

THEOREM 4. *Let* X *be a holomorph-convex space, with* dim X = 2:

i) X *is non degenerate iff* Mani (X) = 2;

ii) X *is a* 1-dim *fibre space over a Stein curve iff* Mani (X) = 1;

iii) X *is compact iff* Mani (X) = 0.

Remark. Theorem 4, 1) has been proved also in [2]. Certainly, Theorem 4 is not true in general if X is not holomorphically convex since there exist non compact **C**-manifolds of dim ≥ 2 with no non constant global holomorphic functions. In conclusion, our previous study said roughly that *generically* holomorph-convex spaces are fibres spaces over Stein spaces.

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