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Twisted Cartesian and Free Products

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Topologia algebrica. — *Twisted Cartesian and Free Products.* Nota di ELYAHU KATZ, presentata (*) dal Corrisp. E. MARTINELLI.

RIASSUNTO. — Vengono posti in relazione i concetti di fibrato principale (p.f.b.) e di cofibrato principale (p.c.b). In tale base vengono ottenuti risultati relativi ai cafibrati principali, partendo dalla ben nota struttura dei fibrati principali.

§ 1. INTRODUCTIONS

Principal co-fiber bundles (p.c.b) were first introduced in [8] for the category of simplicial groups. The main result of the two papers [8] and [9] is a classification theorem of p.c.b's.

The definition of a p.c.b. was extended to the category of topological groups in [5]. There p.c.b's were related to principal fiber bundles by two functors.

In this paper we adapt the above mentioned functors to the category of simplicial sets, and deduce results about p.c.b's from the well known structure of p.f.b's. In particular a theorem of Mienor's $\approx$[7, Theorems 5.1] is extended to a complete classification of p.f.b's via loop homotopy classes of homomorphisms into the fiber. This theorem yields at once the classification of p.c.b's.

A remark about the notations is necessary. We follow the notation of p.c.b's as in [8] and of p.f.b's as in [2]. Kan's construction GX is taken from [2] and not as in [8].

§ 2. THE RELATION BETWEEN P.F.B'S AND P.C.B'S

The relation consists of two functors which are inverses to each other. The categories involved are described next.

DEFINITION 1. A p.f.b $\alpha = \langle S, A, X, p \rangle$ is a group A operating principaly on the left of S, where p is the projection of S on the quotient complex of the operation, X [6]. A morphism between two p.f.b's is a pair $\langle f, g \rangle : \langle S_1, A_1, X, p_1 \rangle \to \langle S_2, A_2, X, p_2 \rangle$ where $f: A_1 \to A_2$ is a homomorphism and $g: S_1 \to S_2$ is a map such that $g(as) = f(a) g(s)$ and $p_1(s) = p_2(g(s))$. Let $\mathcal{F}$ be the category of the p.f.b's and their morphisms.

DEFINITION 2. A p.c.b $\beta = \langle A, T, F(X), \varphi_X, \Psi \rangle$ in a co-group $\langle F(X), \varphi_X \rangle$ with co-basis X [4] which co-operates freely on the group T

(*) Nella seduta del 13 marzo 1976.

via $\Psi : T \to T * F(X)$, with invariant subgroup A. [Let r_1 and r_2 be the projections of $T * F(X)$ on T and $F(X)$ respectively. A free co-coperation means that $(1 * \Psi)\Psi = (\Psi * 1)\Psi$, $r_1 \Psi = 1_T$ and $r_2 \Psi$ is onto. If we consider Γ as a subgroup of $T * F(X)$, then $A = \Psi^{-1}(T)$]. A morphism $\langle f, g\rangle : \langle A_1, T_1, F(X), \varphi_x, \Psi_1\rangle \to \langle A_2, T_2, F(X), \varphi_x, \Psi_2\rangle$ is a homomorphism $g : T_1 \to T_2$ with $f = g|_{A_1} : A_1 \to A_2$, which makes the following diagram commutative:

$$\begin{tikzcd} T_1 \arrow[r, "g"] \arrow[d, "\Psi_1"'] & T_2 \arrow[d, "\Psi_2"] \\ T_1 * F(X) \arrow[r, "g*1"] & T_2 * F(X). \end{tikzcd}$$

The above mentioned p.c.b's and morphisms form the category $\mathscr{C}$.

DEFINITION 3. J is a functor from $\mathscr{C}$ to $\mathscr{F}$, which assigns to β the p.f.b. $J(\beta) = \langle A, T_J, X, \Psi_J\rangle$, where $T_J = \Psi^{-1}(T \times X)$, $\Psi_J = r_2 \Psi|_{T_J} : T_J \to X$, and the operation of A con T_J is to make the following diagram commutative:

$$\begin{tikzcd} A \times T_J \arrow[r, hook] \arrow[d, dotted] & T \times T \arrow[d, "m"] \\ T_J \arrow[r, hook] & T. \end{tikzcd}$$

[$T \times X$ is considered as a subset of $T * F(X)$ and m is the multiplication of T]. The second functor E assings to α a p.c.b.

$E(\alpha) = \langle A, S_E, F(X), \varphi_x, p_E\rangle$, where $S_E = \dfrac{F(S)}{\{a \cdot s = as\}}$ and p_E is defined by the following diagram:

$$\begin{tikzcd} F(S) \arrow[r, "\varphi_S"] \arrow[d, "q"'] & F(S) * F(S) \arrow[r, "1*F(p)"] & F(S) * F(FX) \arrow[d, "q * 1"] \\ S_E \arrow[rr, dotted, "p_E"] & & S_E * F(X). \end{tikzcd}$$

(The element of S "as" is obtained by "a" acting on "s", while $a \cdot s$ is a word of two elements in F(S). The map $q : F(S) \to S_E$ is the quotient homororphism]. The functors J and E are defined on the morphisms in the natural way.

THEOREM 1. *The functors* E *and* J *are inverses to each other. The proof is straightforward.*

§ 3. TWISTED CARTESIAN PRODUCTS (T.C.P.) AND TWISTED FREE PRODUCTS (T.F.P.)

For the readers sake we reproduce the definition of a t.f.p and a matching definition of t.c.p.

DEFINITION 4. A t.c.p. $A \times_t X$ with group A and twisting function $t : X \to A$ of a degree -1, is a complex $A \times X$ whose face and degeneracy operators $\bar{\partial}_i$ and $\bar{s}_i$ are defined as follows:

$$\bar{s}_i(a, x) = (s_i a, s_i x) \quad i \geq 0, \qquad \bar{\partial}_i(a, x) = (\partial_i a, \partial_i x) \quad i > 0,$$

$$\bar{\partial}_0(a, x) = ((\partial_0 a), t(x), \partial_0 x).$$

The twisting function t has to satisfy for $x \in X_n$:

$$1) \quad \partial_0 t(x) = t(\partial_1 x)\, t(\partial_0 x)^{-1}, \qquad 2) \quad t(s_0 x) = 1,$$

$$3) \quad \partial_i t(x) = t(\partial_{i+1} x) \quad i > 0, \qquad 4) \quad s_j t(x) = t(s_{j+1} x) \quad j > 0.$$

A t.f.p. $A *_t F(X)$, with a group A and twisting function $t : X \to A$ of degree -1, is the complex $A * F(X)$ whose degeneracy and boundary operators $\bar{s}_i$ and $\bar{\partial}_i$, are defined as follows on the generators of $A * F(X)$:

$$\bar{\partial}_i a = \partial_i a, \qquad \bar{s}_i a = s_i a, \qquad \bar{s}_i x = s_i x,$$

$$\bar{\partial}_i x = \partial_i x \quad i > 0, \qquad \bar{\partial}_0 x = t(x)\, \partial_0 x.$$

The twisting function is to satisfy 1–4.

We associate every t.c.p. $A \times_t X$ with a p.f.b. $\langle A, A \times_t X, X, p\rangle$ which is also denoted by $A \times_t X$, where $p : A \times_t X \to X$ is the projection. Similarly it is not hard to associate a t.f.p. $A *_t F(X)$ with a p.c.b. which will also be denoted by $A *_t F(X)$.

THEOREM 2. $J(A *_t F(X)) = A \times_t (X)$, $\qquad E(A \times_t X) = A *_t F(X)$.
The proof is ovvious.

p.f.b's and p.c.b's were related to t.c.p's and t.f.p's.

THEOREM A [1]. *Every p.f.b. is a t.c.p.*

THEOREM B [9]. *Every p.c.b. is a t.f.p.*

Both theorems wer proved independently. However as a consequence of Theorems 1 and 2, once one of the theorems is proved, the other follows immediately.

§ 4. THE CLASSIFICATION THEOREMS

In this section X will stand for a complex with only one element in X_0.

DEFINITION 5. Let $f: A \to B$ be a homormorphism.
Then $f*(A\times_t X) = B\times_{ft} X$ is the co-induced p.f.b. from $A\times_t X$ by f, and $f*(A *_t F(X)) = B *_{ft} F(X)$ is the induced p.c.b. from $A *_t F(X)$ by f.

THEOREM 3. $E(f*(\alpha)) = f*(E(\alpha))$, $J(f*(\beta)) = f*(J(\beta))$.
The proof is again straightforward.

DEFINITION 6. Two homomorphisms $f_0, f_1: GX \to A$ are loop homotopic [3] if there exists a homomorphism $H: GX \otimes I \to A$ such that $f_0 = Hi_0, f_1 = Hi_1$, where i_0 and i_1 are the inclusions of GX in $GX \otimes I$.

DEFINITION 7. Two p.c.b's, $\langle A, T_i, F(X), \varphi_X \Psi_i\rangle$ $i = 1, 2$, are equivalent, if there exists a homomorphism $\lambda: T_1 \to T_2$ such that $\lambda|_A = 1_A$ and the following diagram is commutative:

$$\begin{tikzcd} T_1 \arrow[r, "\Psi_1"] \arrow[d, "\lambda"'] & T_2 * F(X) \arrow[d, "\lambda * 1"] \\ T_2 \arrow[r, "\Psi_2"] & T_2 * F(X). \end{tikzcd}$$

Two p.f.b's $\langle A, S_i, X, p_i\rangle$ $i = 1, 2$, are equivalent if there exists a map $\lambda: S_1 \to S_2$ such that $\lambda(as) = a\lambda(s)$, and the following diagram commutes:

$$\begin{tikzcd} S_1 \arrow[r, "\lambda"] \arrow[d, "p_1"'] & S_2 \arrow[d, "p_2"] \\ X \arrow[r, "1"] & X \end{tikzcd}$$

THEOREM 4. *The functors* E *and* J *preserve equivalences. The proof is again easy.*

THEOREM 5. *There is a one-to-one correspondence between loop homotopic classes of homomorphisms from* GX *to* A *and equivalence classes of p.f.b's with base* X *and fiber* A. *The class of the homomorphism of* $f: GX \to A$ *corresponds to the class of the p.f.b.* $A\times_{t\tau} X$, *co-induced by* f *from* $GX\times_\tau X$. [For $\chi \in X_n$, $\tau(x)$ *is the class of* x *in* $(GX)_{n-1}$].

Prof. of Theorem 5. There is a one-to-one correspondence between equivalence classes of p.f.b's $A\times_t X$ and homotopy classes of maps $X \to \overline{W}(A)$.

The correspondence is induced by the function which sends $A\times_t X$ to the map $g_t(x) = [t(x), t(\partial_0 x), \cdots, t(\partial_0^i x), \cdots, t(\partial_0^{n-1} x)] \times \in X_n$.

The isomorphism of the adjoint functors G and $\overline{W}$, which sends $f: GX \to A$ to the map $f': X \to \overline{W}A$, where $f'(x) = [f\tau(x), f\tau(\partial_0 x), \cdots$ $\cdots, f\tau(\partial_0^i x), \cdots, f\tau(\partial_0^{n-1} x)] \times \in X_n$, induces a one-to-one correspondence between loop homotopy classes of homomorphisms and homotopy classes of maps (see [2] or [6]).

Consider the p.f.b. $A \times_{f\tau} X$ co-induced by the homomorphism $f: GX \to A$ from the p.f.b. $GX \times_{\tau} X$. Since $f' = g_{f\tau}$ the proof is completed.

THEOREM 6. [8, 9]. *There is a one-to-one correspondence between loop homotopy classes of homormorphisms* $GX \to A$ *and equivalence class of p.c.b's* $A *_t F(X)$. *This correspondence is induced by the function which assigns to a homomorphism* $f: GX \to A$ *the induced p.c.b.* $A *_{f\tau} F(X)$ *from* $GX *_{\tau} R[X]$.

The proof follows from the following sequence of equivalent statements:

(*a*) β_1 is equivalent to β_2;

(*b*) $J(\beta_1)$ is equivalent to $J(\beta_2)$ (Theorem 4);

(*c*) $J(\beta_1)$ and $J(\beta_2)$ are co-induced by loop homotopic maps from $GX \times_{\tau} X$. (Theorem 5);

(*d*) $EJ(\beta_1)$ and $EJ(\beta_2)$ are induced by loop homotopic maps from $E(GX \times_{\tau} X)$ (Theorem 3);

(*e*) β_1 and β_2 and induced by loop homotopic maps from $GX *_{\tau} FX$. (Theorems 1 and 2).

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