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Affine transformations on Banach manifolds

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Geometria differenziale. — *Affine transformations on Banach manifolds.* Nota di Anastasie Mihai, presentata (*) dal Socio B. Segre.

Riassunto. — In questo lavoro si dimostra che, in determinate condizioni, il gruppo delle trasformazioni affini di una varietà riemanniana di dimensione infinita coincide col gruppo delle isometrie. Un risultato di questo tipo, nel caso della dimensione finita, è stato precedentemente ottenuto da S. Kabayashi [2].

In this paper we prove that, under certain conditions, the group of affine transformations of a Riemannian infinite-dimensional manifold M is equal to the group of isometries of M. A result of the same type, in the finite-dimensional case, has been obtained by S. Kobayashi [2].

1. Affine morphisms of Banach manifolds

We work in the category of infinite-dimensional manifolds of class C^∞. Let M be a Banach manifold. We suppose the existence of a connection map $K : T^2 M \rightarrow TM$ and denote by ∇ the covariant differentiation associated with it, see [1]. For X, Y in $\mathscr{X}$ (M), the F (M)-module of vector fields on M. we set

$$(1.1) \qquad \nabla_X Y = K \circ TY(X) \quad , \quad \frac{Dc}{dt} = K \circ T\dot{c} ,$$

where TY is the tangent map of $Y : M \rightarrow TM$ and $c : [0, 1] \rightarrow M$ a curve on M. The holonomy groups, denoted by $\Phi(p)$, for p in M, were introduced and studied in [4].

Definition 1.1. A Banach manifold M, endowed with a connection map, is said to be irreducible if $\Phi(p)$ does not have any trivial invariant subspace. Otherwise, it is called reducible.

Definition 1.2. Let M and M′ be endowed with the connection maps K and K′, respectively. A morphism $f : M \rightarrow M'$ is called affine if, and only if,

$$(1.2) \qquad Tf \circ K = K' \circ T^2 f.$$

If $M = M'$ and f is a diffeomorphism, we say that f is an affine transformation.

In the following theorem we collect some facts about affine morphisms, needed in the next section; for the proof see [5].

Theorem 1.1 *Let* M *and* M′ *be Banach manifolds with the connection maps* K *and* K′, *respectively. Suppose* $f : M \rightarrow M'$ *is an affine diffeomorphism. Then*:

a) $Tf \circ \tau_c = \tau'_{f \circ c} \circ Tf$ *for every curve* c, *where* τ_c (*resp.* $\tau'_{f \circ c}$) *denotes the parallel displacement along the curve* c (*resp.* $f \circ c$);

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b) $Tf(\nabla_X Y) = \nabla'_{TfX} TfY$, *for all* X, Y *in* $\mathscr{X}(M)$;

c) $Tf \circ R(X, Y) Z = R'(TfX, TfX) TfZ$, *for all* X, Y Z *in* $\mathscr{X}(M)$,

where R (*resp.* R') *denotes the curvature tensor field associated with* K (*resp.* K').

Let (M, g) be a Riemannian manifold. As in the finite dimensional case, the sectional curvature for a 2-plane $\sigma = \{X, Y\}$ in $T_p M$ (the tangent space at p in M) is defined by

$$K_p(\sigma) = \frac{g(R(X, Y) Y, X)}{g(X, Y) g(Y, Y) - g^2(X, Y)} . \tag{1.3}$$

DEFINITION 1.3. Let (M, g) and (M', g') be Riemannian manifolds. A morphism $f: M \to M'$ is called a homothety if

$$g'(TfX, TfY) = c^2 g(X, Y) \quad \text{for any } X, Y \text{ in } \mathscr{X}(M). \tag{1.4}$$

If in (1.4), $c = 1$, then f is an isometry.

It is proved in [1, p. 38] that every isometry is an affine morphism (with respect to the unique connections without torsion defined by g and g', respectively).

In particular, the group of isometries of M is a subgroup of the group of affine transformations of M.

2. THE MAIN RESULTS

The purpose of this section is to prove Theorems 2.1 and 2.2.

THEOREM 2.1. *Let* (M, g) *be an irreducible Riemannian manifold, with bounded and non-identically zero sectional curvature. Then, the group of affine transformations of* (M, g) *is equal to the group of isometries of* (M, g).

Proof. The proof will be given in three steps.

STEP 1. Every homothety is an affine transformation. Using a homothety f, we define a new Riemannian metric on M by $g'(X, Y) = g(TfX, TfY) = c^2 g(X, Y)$. Obviously, $f: (M, g') \to (M, g)$ is an isometry, hence an affine transformation. But, from the definitions of the Riemannian connection [1, p. 36], it follows that the connection defined by g' and g coincide; therefore $f: (M, g) \to (M, g)$ is an affine transformation.

STEP 2. If (M, g) is irreducible, every affine transformation is a homothety. For this we need the following

LEMMA. *Let* H *be a real Hilbert space,* O(H) *the orthogonal group and* S *a subgroup of* O(H) *which acts irreducibly on* H. *If* g *is a symmetric and bilinear form on* H, *invariant under the action of* S, *then there is a constant* c *such that* $g(u, v), = c(u, v,$ *for all* u, v *in* H, (,) *being the standard inner product of* H.

Proof of Lemma. There exists a symmetric operator A such that $g(u, v) = (Au, v)$. Let s be an element of S. From $g(su, sv) = g(u, v)$ (invariance of g) it follows $As = sA$ for all s in S and from Theorem 6, Appendix II of [3], if follows that there exists a constant c, such that $A = cI$ (where I is the identity operator) and therefore $g(u, v) = c(u, v)$. We remark that, if g is positive definite, the constant c must be positive.

We give now the proof of Step 2.

For p in M there are two inner products g_p and g'_p on $T_p M$, where $g'_p(X, Y) = g(T_p f X, T_p f Y)$. As f is an affine transformation g is invariant under the action of $\Phi(p)$ which is a subgroup of the orthogonal group $O(T_p M)$ (with respect to the inner product g). We are in position to apply the Lemma and we obtain $g'_p = c_p^2 g_p$. But g and g' are the parallel tensor fields with respect to the Riemannian connection defined by g, therefore c_p does not depend on p i.e. f is a homothety.

STEP 3. In the hypothesis of Theorem 2.1, every affine transformation is an isometry.

Let f be an affine transformation of M. By Step 2, f is a homothety. If $c = 1$, the proof is complete. Suppose $c < 1$, otherwise we may use f^{-1} and denote by $K < +\infty$ the bound of the sectional curvature. For every p in M and the 2-plane $\sigma = \{X, Y\}$ in $T_p M$ we have

$$|K_p(\sigma)| = c^{2m} \, |K_{f^m(p)}((T_p f)^m X, (T_p f)^m Y)| \leq c^{2m} \cdot K$$

and, for $m \to \infty$, we obtain $K_p(X, Y) \equiv 0$ which is a contradiction.

In the case of M irreducible and complete, the hypothesis "bounded sectional curvature" can be *weakened*. Firstly, we prove

LEMMA 2.1. *Let* (M, g) *be a complete Riemannian manifold. Every strict homothety* (i.e. *with* $c \neq 1$) *of* M, *has a fixed point.*

Proof. (M, g) is a complete metric space with respect to the metric $d(p, q) = \inf\limits_b \left\{ \int_0^1 g(\dot b, \dot b)^{\frac{1}{2}} dt \right\}$ for all curves b on M, with $b(0) = p$ and $b(1) = q$, see [5].

Let f be a homothety with $c < 1$, otherwise we may take f^{-1}. We have

$$d(f(p), f(q)) = \inf \left\{ \int_0^1 g(\dot{\widehat{f \circ b}}, \dot{\widehat{f \circ b}})^{\frac{1}{2}} dt \right\} \leq c \inf \left\{ \int_0^1 g(\dot b, \dot b)^{\frac{1}{2}} dt \right\} \leq c\,d(p, q)$$

therefore f is a contraction map. It follows that f has a fixed point.

Now we give the following

DEFINITION 2.1. The Riemannian manifold (M, g) is said to be with locally bounded sectional curvature if any p in M admits a closed neighbourhood on which the sectional curvature is bounded.

THEOREM 2.2. *Let* (M, g) *be a complete and irreducible Riemannian manifold with locally bounded and non-identically zero sectional curvature. Then, the group of affine transformations of* M *is equal to the group of isometries of* M.

Proof. By Step 2 of the proof of Theorem 2.1, every affine transformation f is a homothety and therefore by Lemma 2.1, has a fixed point, denoted by p_0. Let U be a closed neighbourhood of p_0 on which the sectional curvature is bounded by $K < +\infty$ Suppose $c < 1$ and we have

$$d(p_0, f^m(p)) = d(f^m(p_0), f^m(p)) \leq c^m d(p_0, p);$$

for all p in M; hence there exists an m_0 such that for $m_0 > m$, $f^m(p)$ belongs to U. From

$$|K_p(X, Y)| = c^{2m} |K_{f^m(p)}((T_p f)^m X, (T_p f)^m Y)| \leq c^{2m} \cdot K$$

it follows, when $m \to \infty$, $K_p(X, Y) \equiv 0$ which is a contradiction.

Remark 2.1. The hypothesis of Theorem 2.1 are satisfied by a δ-pinched Riemannian manifold (i.e. there exists a constant $0 < \delta < 1$ such that $\delta < K_p < 1$, for every p in M).

Remark 2.2. When M is a finite-dimensional Riemannian manifold, every p in M admits a neighbourhood such that $\bar{U}$ is compact. As K_p is a continuous function, it follows that it is bounded; therefore M has locally bounded sectional curvature. In the case when M is complete and irreducible, we obtain the theorem by S. Kobayashi ([2]).

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