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Some theorems on Abel type summability methods

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Serie. — *Some theorems on Abel type summability methods.* Nota di BABBAN PRASAD MISHRA e DINESH SINGH, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Gli Autori migliorano alcuni risultati ottenuti in precedenza da uno di essi (B. P. Mishra) relativi alle sommazioni delle serie del tipo di Abel.

1. Following the notations and definitions given in [10], we shall prove the following theorems:

THEOREM 1.1. *If the sequence* $\{S_n^{\alpha}\}$ *is summable* (A_λ) *to* s *and the sequence* $\{na_n\}$ *is summable* (C, β) *to zero,* $\beta \geq 0$, $\beta \geq \alpha > -1$ *and* $\lambda > -1$, *then the sequence* $\{s_n\}$ *is summable* $(C, \beta - 1)$ *to* s.

THEOREM 1.2. *Suppose that* $p \geq 1$, $\beta \geq \alpha > -1$ *and* $\lambda > -1$ *Then, if*

$$\sum_{n=1}^{m} |T_n^{\beta}|^p = o(m) \tag{1.1}$$

and the sequence $\{S_n^{\alpha}\}$ *is summable* (A_λ), *the sequence* $\{s_n\}$ *is summable* $[C, \beta]_p$.

THEOREM 1.3. *If the sequence* $\{S_n^{\alpha}\}$ *is summable* (A_λ) *and the sequence* $\{na_n\}$ *is bounded* (C, β), $\beta \geq \alpha > -1$, $\beta \geq 0$, *then the sequence* $\{s_n\}$ *is summable* $(C, \beta - 1 + \delta)$, $\delta > 0$.

THEOREM 1.4. *If the sequence* $\{S_n^{\alpha}\}$ *is summable* $[A_{\lambda-1}^{*}]_p$ *then the sequence* $\{S_n^{\beta}\}$ *is summable* $[A_{\lambda-1}^{*}]_p$, $p \geq 1$, $\lambda > -1$ *and* $\beta > \alpha > -1$.

THEOREM 1.5. *Let* $p \geq 1$, $\beta \geq \alpha > -1$, $\beta \geq 0$. *Then, if the sequence* $\{S_n^{\alpha}\}$ *is summable* $[A_{\lambda-1}^{*}]_p$ *to* s, $\{s_n\}$ *is summable* $|C, \beta|_p$, *then the sequence* $\{s_n\}$ *is summable* $[C^{*}, \beta]_p$ *to* s.

THEOREM 1.6. *Let* $p \geq 1$, $\beta \geq \alpha > -1$ *and* $\lambda > -1$. *If the sequence* $\{S_n^{\alpha}\}$ *is summable* $|A_\lambda|_p$ *and the sequence* $\{na_n\}$ *is summable* $|C, \beta + 1|_p$, *then the sequence* $\{s_n\}$ *is summable* $|C, \beta|_p$.

THEOREM 1.7. *If the sequence* $\{S_n^{\alpha}\}$ *is summable* $[A_{\lambda-1}]_p$, *then the sequence* $\{S_n^{\beta}\}$ *is summable* $[A_{\lambda-1}]_p$, $p \geq 1$, $\lambda > -1$, $\beta \geq \alpha > -1$.

THEOREM 1.8. *Let* $p \geq 1$, $\beta \geq \alpha > -1$ *and* $\lambda > -1$. *Then, if the sequence* $\{S_n^{\alpha}\}$ *is summable* $[A_{\lambda-1}]_p$ *to the sum* s *and* (1.1) *is satisfied, the sequence* $\{s_n\}$ *is summable* $[C, \beta]_p$ *to* s.

(*) Nella seduta del 13 marzo 1976.

THEOREM 1.9. *Let $p \geq 1$, $\beta > \alpha - 1$ and $\lambda > -1$. Then*

$$\sup_m \left\{ \frac{1}{m+1} \sum_{n=0}^{m} | S_n^{\beta-1} |^p \right\}^{1/p} \leq C(p, \alpha, \beta) \sup_m \left\{ \frac{1}{m+1} \sum_{n=1}^{m} | T_n^{\beta} |^p \right\}^{1/p} +$$

$$+ C \sup_{0 \leq R < 1} \left\{ (1 - R) \int_0^R \frac{| \varphi_\lambda^\alpha(x) |^p}{(1-x)^2} \, dx \right\}^{1/p}.$$

The cases $\lambda = 0$, $\alpha = 0$ of Theorems 1, 3–9 and 2 are due respectively to Flett [2–4] and Kogbetliantz [6] while the cases $\lambda > -1$, $\alpha = 0$ are due to Mishra [8, 9, 10].

2. The lemmas, needed for the proofs of above results, are given below.

LEMMA 2.1. *If $\beta > \alpha > -1$, $\lambda > -1$ and if the series*

$$\varphi_\lambda^\alpha(x) = (1-x)^{\lambda+1} \sum_{n=0}^{\infty} E_n^\lambda S_n^\alpha x^n$$

is convergent for $0 \leq x < 1$, then, for all x in $(0,1)$,

$$\varphi_\lambda^\beta(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha)\,\Gamma(\alpha+1)} (1-x)^{\alpha+1} \cdot x^{-\beta} \int_0^x (x-t)^{\beta-\alpha-1} \cdot (1-t)^{-\beta-1} \cdot t^\alpha \varphi_\lambda^\alpha(t) \, dt.$$

The above lemma has been stated by Kuttner in [7]. The proof of this lemma runs parallel to that of Lemma 1 of one of the authors (Mishra [10]).

LEMMA 2.2. *Let $\alpha > -1$, $\lambda > -1$. Then, if $\varphi_\lambda^\alpha(x)$ converges for $|x| < 1$ and*

$$\varphi_\lambda^{\alpha,1}(x) = (1-x)^{\lambda+1} \sum_{n=0}^{\infty} E_n^\lambda S_n^{\alpha,1} x^n,$$

where

$$S_n^{\alpha,1} = (E_n^{\alpha+1})^{-1} \sum_{v=0}^{n} E_{n-v}^{\alpha-1} E_v^1 s_v;$$

$$\varphi_\lambda^{\alpha,1}(x) = (\lambda+1)\, \varphi_{\lambda+1}^{\alpha+1}(x) - \lambda \varphi_\lambda^{\alpha+1}(x).$$

Proof. Since the convergence of $\varphi_\lambda^\alpha(x)$ and $\Phi_\lambda^{\alpha,1}(x)$ are equivalent, we have, for $|x| < |$.

$$\varphi_\lambda^{\alpha,1}(x) = (1-x)^{\lambda+1} (\alpha+1) \sum_{n=0}^{\infty} s_n (n+1) x^n \sum_{v=0}^{\infty} \frac{E_{v+n}^\lambda}{E_{v+n}^\alpha} E_v^{\alpha-1} \frac{x^v}{(\alpha+n+v+1)} =$$

$$= (1-x)^{\lambda+1} \sum_{n=0}^{\infty} \frac{\alpha+1}{\alpha} s_n x^n \sum_{v=0}^{\infty} \frac{E_{v+n}^\lambda}{E_{v+n}^\alpha} E_v^{\alpha-1} x^v \left\{ \frac{(v+\alpha)(n+v+1)}{(\alpha+n+v+1)} - v \right\} =$$

$$= (1-x)^{\lambda+1} \sum_{n=0}^{\infty} s_n x^n \sum_{v=0}^{\infty} \frac{E_{v+n}^{\lambda}}{E_{v+n}^{\alpha}} E_v^{\alpha-1} \frac{(\alpha+1)(v+\alpha)\{(1-x)(\lambda+n+v+1)-\lambda\}}{\alpha(\alpha+n+v+1)} x^v =$$

$$= (1-x)^{\lambda+2} \sum_{n=0}^{\infty} s_n x^n \sum_{v=0}^{\infty} \frac{E_{v+n}^{\lambda}}{E_{v+n}^{\alpha}} E_v^{\alpha-1} \frac{(\alpha+1)(v+\alpha)(\lambda+n+v+1)}{\alpha(\alpha+n+v+1)} x^v -$$

$$-\lambda(1-x)^{\lambda+1} \sum_{n=0}^{\infty} s_n x^n \sum_{v=0}^{\infty} \frac{E_{v+n}^{\lambda}}{E_{v+n}^{\alpha}} E_v^{\alpha-1} \frac{(\alpha+1)(v+\alpha)}{\alpha(\alpha+n+v+1)} x^v =$$

$$= (\lambda+1)(1-x)^{\lambda+2} \sum_{n=0}^{\infty} s_n x^n \sum_{v=0}^{\infty} \frac{E_{v+n}^{\lambda+1}}{E_{v+n}^{\alpha+1}} E_v^{\alpha} x^v -$$

$$-\lambda(1-x)^{\lambda+1} \sum_{n=0}^{\infty} s_n x^n \sum_{v=0}^{\infty} \frac{E_{v+n}^{\lambda}}{E_{v+n}^{\alpha+1}} E_v^{\alpha} x^v =$$

$$= (\lambda+1)\varphi_{\lambda+1}^{\alpha+1}(x) - \lambda\varphi_{\lambda}^{\alpha+1}(x).$$

Hence the lemma is completely established.

LEMMA 2.3. *Let* $\alpha > -1$ *and* $\lambda > -1$. *Then the* (A_λ)*-summability of the sequence* $\{S_n^{\alpha}\}$ *implies the* $(A_{\lambda+1})$*-summability of* $\{S_n^{\alpha+1}\}$.

Proof. If we assume $S_n^{\alpha,1}$ as $(C, \alpha, 1)$ mean of the sequence $\{s_n\}$ then clearly the $(C, \alpha, 1)$ and (C, α) methods are equivalent. Now the proof of Lemma 2.3 follows from this remark, Lemma 2.1 with $\beta = \alpha + 1$ and Lemma 2.2.

LEMMA 2.4. ([5], Theorem 319(b)). *Suppose that* $p > 1$, *that* $K(x, y)$ *is non-negative and homogeneous of degree* -1 *and that*

$$\int_0^{\infty} K(x, 1)\, x^{-1/p}\, dx = \int_0^{\infty} K(1, y)\, y^{-1/p'}\, dy = C$$

where p' *denotes a number conjugate to* $p > 1$ *in Hölder's inequality. Then*

$$\int_0^{\infty} dy \left| \int_0^{\infty} K(x, y) f(x)\, dx \right|^p \leq C^p \int_0^{\infty} |f(x)|^p\, dx.$$

3. In this section, we consider the proofs of the theorems mentioned in Section 1. After Lemmas 2.1 and 2.3, the proofs of Theorems 1.1, 1.2 and 1.3 run parallel to those of Theorems 2, 3 and 4 of Mishra [10]. To prove Theorem 1.4, it is enough to prove that

$$\int_0^1 \frac{|\varphi_\lambda^{\beta}(x)|^p}{(1-x)}\, dx \leq \int_0^1 \frac{|\varphi_\lambda^{\alpha}(x)|^p}{(1-x)}\, dx, \tag{3.1}$$

for $p \geq 1$, $\beta \geq \alpha > -1$ and $\lambda > -1$.

Let us suppose without any loss of generality that $s = 0$. Then (3.1) reduces to

$$(3.2)\qquad \int_0^1 \frac{|\varphi_\lambda^\beta(x)|^p}{x(1-x)}\,dx \le \int_0^1 \frac{|\varphi_\lambda^\alpha(x)|^p}{x(1-x)}\,dx.$$

After the transformation $x = y(1+y)^{-1}$, (3.2) becomes

$$(3.3)\qquad \int_0^\infty \frac{|\sigma_\lambda^\beta(y)|^p}{y}\,dy \le \int_0^\infty \frac{|\sigma_\lambda^\alpha(y)|^p}{y}\,dy,$$

and Lemma 2.1 takes the form

$$\varphi_\lambda^\beta\left(\frac{y}{1+y}\right) = \sigma_\lambda^\beta(y) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha)\,\Gamma(\alpha+1)}\, y^{-\beta}\int_0^y (y-x)^{\beta-\alpha-1} x^\alpha \sigma_\lambda^\alpha(x)\,dx,$$

We now apply Lemma 2.4 with

$$f(x) = \sigma_\lambda^\alpha(x)\cdot x^{-1/p}$$

and

$$K(x,y) = \begin{cases} y^{-(1/p)-\beta}\cdot x^{\alpha+(1/p)}\cdot(y-x)^{\beta-\alpha-1}, & x<y, \\ 0, & x \ge y. \end{cases}$$

and get the desired result.

The proofs of Theorems 1.5 and 1.6 are similar, after (3.1), to those of Theorems 5 and 6 of Mishra [10].

The proof of Theorem 1.7 is concerned with the inequality

$$(3.4)\qquad \int_0^R \frac{|\varphi_\lambda^\beta(x)|^p}{(1-x)^2}\,dx \le C(p,\beta,\alpha)\int_0^1 \frac{|\varphi_\lambda^\alpha(x)|^p}{(1-x)^2}\,dx, \qquad 0<R<1.$$

Making the same change of variable as in (3.2) and writing $Y = \dfrac{R}{1-R}$, (3.4) is equivalent to

$$(3.5)\qquad \int_0^Y |\sigma_\lambda^\beta(y)|^p\,dy \le C\int_0^Y |\sigma_\lambda^\alpha(y)|^p\,dy.$$

We now take

$$f(x) = \begin{cases} \sigma_\lambda^\alpha(x), & x \le y, \\ 0, & x > y \end{cases}$$

and

$$K(x,y)=\begin{cases} y^{-\beta}\cdot(y-x)^{\beta-\alpha-1}\cdot x^{\alpha}\ , & x<y \\ 0\ , & x\geq y \end{cases}$$

in Lemma 2.4, we see that

$$\int_0^{\infty} |\sigma_\lambda^\beta(y)|^p\,dy \leq C\int_0^{\infty} |\sigma_\lambda^\alpha(y)|^p\,dy.$$

Let

$$g(y)=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha)\,\Gamma(\alpha+1)}\,y^{-\beta}\int_0^{y}(y-x)^{\beta-\alpha-1}\cdot x^{\alpha}f(x)\,dx.$$

Then, clearly

$$\sigma_\lambda^\beta(y)=g(y)\qquad \text{for}\quad y\leq Y.$$

Hence

$$\int_0^{Y} |\sigma_\lambda^\beta(y)|^p\,dy \leq \int_0^{\infty}|g(y)|^p\,dy \leq C\int_0^{\infty}|f(y)|^p\,dy = C\int_0^{Y}|\sigma_\lambda^\alpha(y)|^p\,dy.$$

This establishes the inequality (3.5). Theorem 1.7 is an immediate consequence of (3.4). The proof of Theorem 1.8 runs parallel to that of Theorem 7 [10]. An argument similar to that of Theorem 1.8 establishes Theorem 1.9 which is the corresponding result for boundedness.

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