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Oscillations induced by forcing functions

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Equazioni differenziali ordinarie. — *Oscillations induced by forcing functions.* Nota di DAVID LOWELL LOVELADY, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si dànno condizioni sufficienti che assicurano il carattere oscillatorio delle eventuali soluzioni limitate dall'equazione

$$u^{(n)} + (-1)^{n+1} qu = f.$$

Let q be a continuous function from $[0, \infty)$ to $(0, \infty)$, and let n be an integer, $n \geq 2$. It is wellknown (see, for example, [1, Corollary 2.2, p. 508]) that there is a bounded positive solution of

$$u^{(n)} + (-1)^{n+1} qu = 0$$

on $[0, \infty)$. In the present work we shall obtain conditions on a continuous function f which ensure that every bounded solution of

$$u^{(n)} + (-1)^{n+1} qu = f \tag{1}$$

is oscillatory, i.e., has an unbounded set of zeros. Several Authors have recently dealt with nonhomogeneities in oscillation problems (see, for example, [2], [3], [5], and [6]), but most of these studies have considered maintaining oscillation rather than inducing oscillation.

If b is a real number, let $b^+ = (|b| + b)/2$ and $b^- = (|b| - b)/2$. The following theorem is our main result.

THEOREM. *Suppose there is a bounded oscillatory function* φ *on* $[0, \infty)$ *such that* $b > 0 > c$, *where* $b = \limsup_{t \to \infty} \varphi(t)$ *and* $c = \liminf_{t \to \infty} \varphi(t)$, *such that*

$$\int_0^\infty s^{n-1} q(s) (b - \varphi(s))^+ \, ds = \infty \quad \text{and} \quad \int_0^\infty s^{n-1} q(s) (c - \varphi(s))^- \, ds = \infty, \tag{2}$$

and such that $\varphi^{(n)} = f$. *Then every bounded solution of* (1) *is oscillatory.*

Note that our theorem does not guarantee that (1) has an oscillatory solution, for (1) might not have a bounded solution. The equation

$$u''(t) - u(t) = e^t \cos(e^t) - e^{2t} \sin(e^t), \tag{3}$$

with φ given by $\varphi(t) = \sin(e^t)$, has no bounded solutions, as can be seen by elementary means. There are, however, some unbounded oscillatory solu-

(*) Nella seduta del 13 marzo 1976.

tions of (3), so we leave open the question: Do the hypotheses of the theorem guarantee that (1) has an oscillatory solution?

Condition (2) prevents the inequality

$$\int_0^\infty s^{n-1} q(s)\, ds < \infty. \tag{4}$$

This is not too stringent, for if (4) holds then (1) has a bounded positive solution. To see this, suppose (4) holds, and let $\beta > -\inf_{t \geq 0} \varphi(t)$. Now

$$u(t) = \beta + \varphi(t) + \frac{1}{(n-1)!} \int_t^\infty (s-t)^{n-1} q(s)\, u(s)\, ds \tag{5}$$

can be solved by iteration, and the solution of (5) is a positive solution of (1).

Proof of the Theorem. Let u be a bounded nonoscillatory solution of (1). Since $-f$ satisfies the same hypotheses as does f, it suffices to assume u is eventually positive. Find $\beta \geq 0$ such that $u(t) > 0$ if $t \geq \beta$. Let $v = u - \varphi$. Now v is bounded and

$$v^{(n)} + (-1)^{n+1} qu = 0. \tag{6}$$

From (6), $v^{(n)}$ is eventually one-signed, so each $v^{(k)}$, $k = 0, \cdots, n$, is eventually one-signed. Find $\gamma \geq \beta$ such that none of $v, v', \cdots, v^{(n)}$ has a zero in $[\gamma, \infty)$. Since v is bounded, $v^{(k)} v^{(k+1)} < 0$ on $[\gamma, \infty)$, for $k = 0, \cdots, n-1$, and (6) says $v^{(n)} > 0$ if n is even and $v^{(n)} < 0$ if n is odd, so $v^{(k)} > 0$ on $[\gamma, \infty)$ if k is even and $v^{(k)} < 0$ on $[\gamma, \infty)$ if n is odd. Now $v^{(k)}(\infty) = \lim_{t \to \infty} v^{(k)}(t)$ exists for $k = 0, \cdots, n-1$, and $v^{(k)}(\infty) = 0$ if $k = 1, \cdots, n-1$. The possibility $v(\infty) > 0$ is not excluded. Now

$$v(t) = v(\infty) + \frac{1}{(n-1)!} \int_t^\infty (s-t)^{n-1} q(s)\, u(s)\, ds \tag{7}$$

if $t \geq \gamma$ (compare [4, Lemma 2]). If $v(\infty) < -c$, then u is clearly oscillatory, so $v(\infty) \geq -c$. Recall that $v' < 0$, so $v(t) \geq -c$ if $t \geq \gamma$. Now $u(t) \geq \varphi(t) - c$ if $t \geq \gamma$. Since $u > 0$ on $[\gamma, \infty)$, this says $u(t) \geq (\varphi(t) - c)^+ = (c - \varphi(t))^-$ if $t \geq \gamma$. This and (7) say

$$\int_\gamma^\infty (s-\gamma)^{n-1} q(s)\, (c - (s))^- \, ds < \infty,$$

contradicting (2). The proof is complete.

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