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On mappings contractive in the sense of Kannan

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Analisi funzionale. — *On mappings contractive in the sense of Kannan*. Nota di LUDVIK JANOS, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Se $f: X \to X$ è un'applicazione continua compatta di uno spazio metrico (X, d) in sè stesso ed f ha la proprietà che se $x, y \in X$, $x \neq y$ implica che $d(f(x), f(y)) < \frac{1}{2}[d(x, f(x)) + d(y, f(y))]$, allora f ha un unico punto fisso e inoltre f è una contrazione di Banach rispetto a ad un'opportuna metrizzazione dello spazio X.

1. INTRODUCTION

A fixed point theorem for a certain class of metric spaces (X, d) (e.g. complete, bounded, compact, etc.) is an assertion about a selfmap $f: X \to X$ satisfying certain conditions, some of which may be of topological nature (e.g. continuity), and others involving explicitly the metric d. The purpose of this note is to investigate a relation between the classical Banach contraction principle and its generalizations by Kannan (see [4] and [5]) and Edelstein ([2] and [1]). The pertaining metric conditions mentioned above are the three types of contraction inequalities contained in the following definitions: A selfmap $f: X \to X$ of a metric space (X, d) is said to be a *contraction* (in the sense of Banach) if there exists $c \in (0, 1)$ (the Lipschitz constant) such that for all $x, y \in X$ there holds $d(f(x), f(y)) \leq c d(x, y)$. Similarly, we say that f is *Kannan contractive* if $x \neq y$ implies

$$d(f(x), f(y)) < \tfrac{1}{2}[d(x, f(x)) + d(y, f(y))]$$

and *Edelstein contractive* if $x \neq y$ implies

$$d(f(x), f(y)) < d(x, y).$$

Following the idea of Ira Rosenholtz [8] exposed in his paper " Evidence of a conspiracy among fixed point theorems " who observed that one kind of contractivity relative to a metric d may imply the other one relative to a suitable metric d^* topologically equivalent to d, we first focus our attention on the relation between Kannan-contractivity and the Banach contraction property. We denote by $M(X)$ the set of all metrics on X topologically equivalent to d for a given metric space (X, d). Our main result reads as follows:

THEOREM 1.1 *Let $f: X \to X$ be a Kannan contractive continuous and compact mapping of a metric space (X, d) into itself. Then f has a unique fixea point and furthermore for any $c \in (0, 1)$ there exists a metric $d^* \in M(X)$ relative to which f is a contraction with Lipschitz constant c.*

(*) Nella seduta del 13 marzo 1976.

2. Proof of the theorem

Denoting by Y a compact subset of X which according to our assumption contains $f(X)$ we observe that Y is f-invariant, i.e., $f(Y) \subset Y$. This in turn implies that the set A defined as the intersection $\bigcap_{n=1}^{\infty} f^n(Y)$ of the iterative images $f^n(Y)$ is a nonempty compact f-invariant subset of X which is mapped by f onto itself. We now show that A is a one point set, i.e., $A = \{a\}$ for some $a \in X$ which is the unique fixed point of f. If this were not the case then the diameter $d = d(A)$ of A would be a positive number $d > 0$. Due to the compactness of A there would exist points $x, y \in A$ whith $d(x, y) = d$ and due to the fact that f maps A onto itself there would exist points $x^*, y^* \in A$ with $x = f(x^*)$ and $y = f(y^*)$. Applying to these points the Kannan inequality we obtain:

$$d = d(x, y) = d[f(x^*), f(y^*)] < \tfrac{1}{2}[d(x^*, x) + d(y^*, y)] \leq d$$

which is the desired contraction, proving that f has a unique fixed point a.

In order to prove the rest of Theorem 1.1. we need the following result due to P. R. Meyers [7]:

THEOREM 2.1. (P. R. Meyres). *Let $f: X \to X$ be a continuous selfmap of a metric space (X, d) into itself with the following properties*:

(i) *f has a unique fixed point $a \in X$.*

(ii) *for every $x \in X$ the sequence of iterations $\{f^n(x)\}$ converges to a.*

(iii) *there exists an open neighborhood U of a with the property that given any open set V containing a there exists an integer n_0 such that $n \geq n_0$ implies $f^n(U) \subset V$.*

Then for an arbitrary $c \in (0, 1)$ there exists a metric $d^* \in M(X)$ relative to which f is a contraction with the Lipschitz constant c.

We now verify readily that our map f satisfies all of these three conditions of Theorem 2.1. In fact, (i) has already been proved, and (ii) follows from the fact that for any $x \in X$ the sequence $\{f^n(x)\}$ is contained in Y. To show that f satisfies also (iii) we take $U = X$ and observe that $f^{n+1}(X)$ is contained in $f^n(Y)$, the diameter of which diminishes to zero as $n \to \infty$. Thus $f^n(X)$ squeezes into any neighborhood of a and the proof of Theorem 1.1. is complete.

What we have just proved means that Kannan contractivity implies under certain additional conditions and under a suitable remetrisation of the space X the Banach contraction property. We now show that the converse is also true.

LEMMA 2.2. *Let $f: X \to X$ be a contraction of a metric space (X, d) with the Lipschitz constant $c < 1/3$.*

Then f is Kannan contractive relative to d.

Proof. Let $x, y \in X$. From the triangle inequality we obtain:

$$d(x, y) \leq d(x, f(x)) + d(f(x), f(y)) + d(f(y), y)$$

which combined with the contraction inequality $d(f(x), f(y)) \leq cd(x, y)$ yields $d(f(x), f(y)) \leq \frac{c}{c-1}[d(x, f(x)) + d(y, f(y))]$ from which our assertion follows.

THEOREM 2.3. *Let $f: X \to X$ be a contraction of a metric space (X, d) having a fixed point $a \in X$. (We do not postulate (X, d) to be complete).*
Then there exists a metric $d^* \in M(X)$ relative to which f is Kannan contractive.

Proof. Since f has the fixed point $a \in X$, f satisfies all the requirements of Theorem 2.1. Thus, choosing $c < 1/3$ we obtain a new metric $d^* \in M(X)$ relative to which f is a contraction with the Lipschitz constant $< 1/3$ which by Lemma 2.2. implies that f is Kannan contractive relative to d^*.

3. SOME GENERALIZATIONS TO CONDENSING MAPPINGS

Denoting by $\alpha(Y)$ the Kuratowski measure of noncompactness of a subset Y of a metric space (X, d) (see [6] and [9]) we say that a continuous selfmap $f: X \to X$ of a bounded metric space (X, d) is *condensing* if for any nonempty and non-totally bounded subset Y of X there holds $\alpha(f(Y)) < \alpha(Y)$. We note that for our purposes the Kuratowski measure of noncompactness can be replaced by that of Hausforff (see [9]). We now show that if the compactness condition in Theorem 1.1. is relaxed requiring only the mapping f to be condensing and imposing at the same time the condition of completeness on the space (X, d) then the mapping f has a unique fixed point and is Edelstein contractive relative to a suitable metric $d^* \in M(X)$. To prove this statement we need the following result on Edelstein contractive mappings proved in [3].

THEOREM 3.1. *Let X be a metrizable topological space and $f: X \to X$ a continuous map such that the sequence $\{f^n(x)\}$ converges for every $x \in X$. Then the following two statements are equivalent:*

(i) *There is a metric $d \in M(X)$ such that f is Edelstein contractive relative to d.*

(ii) *For every nonempty compact f–invariant subset Y of X the intersection $\bigcap_{n=1}^{\infty} f^n(Y)$ is a one-point set.*

We are now ready to prove.

THEOREM 3.2. *Let f be a condensing Kannan selfmap of a complete bounded metric space* (X, d). *Then f has a unique fixed point and there is a metric* $d^* \in M(X)$ *relative to which f is Edelstein contractive.*

Proof. If $x \in X$ we denote by $\Gamma(x)$ the closure of its orbit $o(x) = \{x, f(x), f^2(x) \cdots\}$. Due to the completeness of (X, d) and the obvious fact that $\alpha[o(f(x))] = \alpha[o(x)]$ we conclude that $\Gamma(x)$ is compact. Since $\Gamma(x)$ is f-invariant we can apply Theorem 1.1. to it obtaining that the sequence $\{f^n(x)\}$ converges to the unique fixed point of f. Thus the hypothesis of Theorem 3.1. is satisfied by f and all we need is to verify that the condition (ii) in that theorem is also satisfied. But this follows by the same argument (using the Kannan contractivity of f) as used in the proof of Theorem 1.1. when showing that $\bigcap_{n=1}^{\infty} f^n(Y)$ is a one–point set, and our assertion follows.

Remark 3.1. We do not know so far whether Theorem 3.2. has a converse in the sense that the contractivity conditions in the sense of Kannan and Edelstein appearing in its hypothesis and conclusion can interchange their rôles. Also we do not know whether the conclusion of this theorem can be strengthened claiming that f is a contraction relative to some metric $d^* \in M(X)$. However, assuming (X, d) compact and $f : X \to X$ continuous we obtain as a corollary of Theorems 1.1, 2.1, 2.3 and 3.1 the equivalence of the following statements:

(i) f is Kannan contractive relative to some $d^* \in M(X)$.

(ii) f is Edelstein contractive relative to some $d^* \in M(X)$

(iii) given $c \in (0, 1)$ there is $d^* \in M(X)$ relative to which f is a contraction with Lipschitz constant c.

Remark 3.2. Theorem 1.1 remains valid if the Kannan contractivity appearing in its hypothesis is replaced by some other types of contractivity as for example introduced by

$$x \neq y \Rightarrow d(f(x), f(y)) < \frac{1}{2}[d(x, f(y)) + d(y, f(x))]$$

or by

$$x \neq y \Rightarrow d(f(x), f(y)) < \frac{1}{3}[d(x, f(x)) + d(x, y) + d(y, f(y))],$$

since the argument in the proof of this theorem concerning the set $A = \bigcap_{n=1}^{\infty} f^n(Y)$ remains valid in these cases.

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