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Giuseppe Da Prato, Mimmo Iannelli, Luciano Tubaro

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Analisi matematica. — *Some results on a stochastic differential equation* (*). Nota di GIUSEPPE DA PRATO, MIMMO IANNELLI e LUCIANO TUBARO, presentata (**) dal Corrisp. G. STAMPACCHIA.

RIASSUNTO. — Si danno alcuni risultati di esistenza e unicità della soluzione per una equazione differenziale stocastica, in condizioni di non lipschitzianeità.

1. INTRODUCTION

The stochastic differential equation:

$$\mathrm{d}u = f(u)\,\mathrm{d}t + g(u)\,\mathrm{d}w \tag{1}$$

has been studied by Ito ([1]) under the assumption that $f: \mathbf{R} \to \mathbf{R}$ and $g: \mathbf{R} \to \mathbf{R}$ are Lipschitz continuous. After [1] this assumption has been weakened by various Authors ([2]–[6]); generally speaking the conditions involved in these papers correspond to the fact that the mappings $u \to f \circ u$ and $u \to g \circ u$ map L^2 into L^2.

In this Note we give some results for (1) under some hypotheses which in the case $g \equiv 0$ are equivalent to suppose f maximal decreasing. These results apply to the problem with $f(x) = -|x|^m \operatorname{sig}(x)$ and $g(x) = x^n$ where $n \geq 2n$. The proofs, which will be given in detail in a later work, are carried through with the techniques of the theory of non-linear semi-groups ([7]); it seems that the results can be extended to Hilbert spaces.

2. THE PROBLEM

Let $(\Omega, \mathscr{E}, P)$ be a probability space, $\{w(t), t \in [0, T]\}$ a Wiener process in $(\Omega, \mathscr{E}, P)$ and $\{\mathscr{F}_t, t \in [0, T]\}$ a family of σ-algebras non anticipating with respect to $w(t)$ (see [2]). We consider the following problem:

$$\text{(P)} \qquad \begin{cases} \mathrm{d}u(t) = f(u(t))\,\mathrm{d}t + g(u(t))\,\mathrm{d}w \\ u(0) = u_0 \in L^2(\Omega, \mathscr{F}_0, P) \end{cases} \qquad t \in [0, T]$$

where $f: \mathbf{R} \to \mathbf{R}$ and $g: \mathbf{R} \to \mathbf{R}$ are measurable functions on which we make more precise hypotheses in the following. We want to solve problem (P) in the sense specified by the following definition:

DEFINITION 1. $u \in C([0, T]; L_t^2)$ *is a solution of* (P) *if it is* $t \to f(u(t)) \in L^2([0, T]; L_t^2)$, $t \to g(u(t)) \in L^2([0, T]; L_t^2)$ *and the conditions in* (P) *are verified.*

(*) Lavoro eseguito nell'ambito del G.N.A.F.A. del C.N.R.
(**) Nella seduta del 13 marzo 1976.

In Definition 1 it is:

$$C([0,T];L_t^2) = \{u \in C([0,T];L^2(\Omega,\mathscr{E},P))\,;\ u(t) \text{ is } \mathscr{F}_t\text{-measurable}\}$$

$$L^2([0,T];L_t^2) = \{u \in L^2([0,T];L^2(\Omega,\mathscr{E},P)\,;\ u(t) \text{ is } \mathscr{F}_t\text{-measurable}\}.$$

2. MAIN HYPOTHESIS AND UNIQUENESS

Our main hypothesis is contained in the following definition

DEFINITION 2. *The pair of functions* (f,g) *is said* α*-decreasing if it is*:

$$(2) \qquad 2(x-y)(f(x)-f(y)) + (g(x)-g(y))^2 \le \alpha(x-y)^2 \qquad x,y \in \mathbf{R}.$$

We remark that if (f,g) is α-decreasing, then the function $x \to f(x) - \frac{\alpha}{2}x$ is decreasing. Definition 2 leads to the following result

THEOREM 1. *Let* u *and* v *be solutions of problem* (P) *with initial data* u_0 *and* v_0 *respectively. If the pair* (f,g) *is* α*-decreasing then it is*:

$$(3) \qquad \|u(t)-v(t)\|_{L^2} \le \exp\left(\frac{\alpha}{2}t\right)\|u_0-v_0\|_{L^2} \qquad t \in [0,T].$$

Obviously (3) yields uniqueness.

3. THE APPROACHING PROBLEM: EXISTENCE

Suppose now that:

(4) *There exist the functions* $J_n : \mathbf{R} \to \mathbf{R}$, $n \in \mathbf{N}$, *such that putting* $f_n(x) = f(J_n(x))$, $g_n(x) = g(J_n(x))$ *the functions* $f_n : \mathbf{R} \to \mathbf{R}$ *and* $g_n : \mathbf{R} \to \mathbf{R}$ *are Lipschitz continuous*

then we can consider the following problem:

$$(P_n) \qquad \begin{cases} du_n(t) = f_n(u_n(t))\,dt + g_n(u_n(t))\,dw \\ u_n(0) = u_0 \end{cases}$$

which has a unique solution in the sense of Definition 1 (this is the Ito solution, see [2]). We then have:

THEOREM 2. *Suppose that* (4) *holds and that the following assumptions are fulfilled*:

$$(5) \qquad |J_n(x) - x| \le \varepsilon(n)\,|f_n(x)| \quad , \quad \varepsilon(n) \xrightarrow[n\to\infty]{} 0$$

$$(6) \qquad f_n(x)\,f_n''(x) \in L^\infty(\mathbf{R})$$

$$(7) \qquad 2f_n'(x)\,f_n^2(x) + (f_n(x)\,f_n'(x))'\,g_n^2(x) \le \beta f^2(x) + \gamma$$

(β *and* γ *are independent of* n).

(8) *The pair* (f, g) *is* α*-decreasing and* $x \to f(x) - \frac{\alpha}{2} x$ *is maximal as a decreasing function*

then it is:

(9) $$u_n \to u \quad in \quad C([0, T]; L_t^2])$$

and u *is the solution of problem* (P).

The conditions (6) and (7) of Theorem 2 are used to get the estimate:

$$\|f_n(u_n(t))\|_{L^2} \leq K \qquad \text{(K independent of } n)$$

on the solution of (P_n) and it is implicitly intended that it must be $f_n \in C^2(\mathbf{R})$.

4. An example

Concerning the example cited in the introduction:

$$f(x) = -|x|^m \operatorname{sig}(x) \qquad g(x) = x^n \qquad m \geq 2n$$

it is easy to show that (f, g) is α-decreasing for some α. As for the hypothesis on the approaching problem take:

$$J_k(x) = \frac{x}{\left(1 + \frac{|x|^{m-1}}{k}\right)^{1/m}}$$

then assumptions (5)–(8) are fulfilled.

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