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# Rendiconti

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## Absolutely A — p summing operators

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**Analisi funzionale.** — *Absolutely* A — *p* *summing operators.* Nota di B. E. RHOADES, presentata [(*)] dal Socio G. SANSONE.

RIASSUNTO. — L'Autore definisce alcune classi di operatori A — *p* sommabili e ne prova alcune proprietà.

DEFINITION. Let E and F be normed spaces. An operator $T : E \to F$ will be called an absolutely A — $p$ summing operator, $1 \leq p < \infty$, if, for each finite sequence $\{x_n\} \subset E$, there exists a constant $C > 0$ such that

$$(1) \qquad \left(\sum_{n=1}^{k}\left(\sum_{j=1}^{\infty} a_{nj} \| Tx_j \|\right)^p\right)^{1/p} \leq C \sup_{\|a\| \leq 1} \left(\sum_{n=1}^{k}\left(\sum_{j=1}^{\infty} a_{nj} \mid \langle x_j , a \rangle \mid\right)^p\right)^{1/p}.$$

where A is a matrix with nonnegative entries such that the sum on the right hand side of (1) is finite. The smallest such constant such that (1) is satisfied will be denoted by $\pi_{A,p}(T)$.

A corresponding definition can be made for a matrix A, which is not nonnegative, by replacing each $a_{nj}$ in (1) with $| a_{nj} |$.

PROPOSITION 1. *For each fixed* A, *the set of absolutely* A — $p$ *summing operators is a normed space under the norm* $\pi_{A,p}$.

*Proof.* Let S , T be absolutely A — $p$ summing operators. Using Minkowski's inequality,

$$\left(\sum_{n=1}^{k}\left(\sum_{j=1}^{\infty} a_{nj} \| (S+T) x_j \|\right)^p\right)^{1/p} \leq$$

$$\leq \left(\sum_{n=1}^{k}\left(\sum_{j=1}^{\infty} a_{nj} \| Sx_j \|\right)^p\right)^{1/p} + \left(\sum_{n=1}^{k}\left(\sum_{j=1}^{\infty} a_{nj} \| Tx_j \|\right)^p\right)^{1/p} \leq$$

$$\leq (\pi_{A,p}(S) + \pi_{A,p}(T)) \sup_{\|a\| \leq 1} \left(\sum_{n=1}^{k}\left(\sum_{j=1}^{\infty} a_{nj} \mid \langle x_j , a \rangle \mid\right)^p\right)^{1/p} <$$

$$< \infty .$$

Moreover, $\pi_{A,p}(S+T) \leq \pi_{A,p}(S) + \pi_{A,p}(T)$.

PROPOSITION 2. *For each fixed* $A \neq 0$, *every absolutely* A — $p$ *summing operator is bounded, and* $\| T \| \leq \pi_{A,p}(T)$.

*Proof.* Since $A \neq 0$, there exists an integer $j$ such that $a_{nj} > 0$ for some integer $n$. For this choice of $j$, set $x_j = x$ , $x_n = 0$ , $n \neq j$. Then (1) becomes

$$\left(\sum_{n=1}^{k} (a_{nj} \| Tx \|)^p\right)^{1/p} \leq \pi_{A,p}(T) \sup_{\|a\| \leq 1} \left(\sum_{n=1}^{k} a_{nj} \mid \langle x , a \rangle \mid)^p\right)^{1/p},$$

(*) Nella seduta del 14 febbraio 1976.

which reduces to

$$\| Tx \| \left(\sum_{n=1}^{k} a_{nj}^{p}\right)^{1/p} \leq \pi_{A,p}(T) \sup_{\|a\|\leq 1} |\langle x , a\rangle| \left(\sum_{n=1}^{k} a_{nj}^{p}\right)^{1/p}.$$

Thus $\| Tx \| \leq \pi_{A,p}(T) \| x \|$; i.e., $\| T \| \leq \pi_{A,p}(T)$.

PROPOSITION 3. *If* F *is a Banach space, then the set of absolutely* $A - p$ *summing operators, for each fixed* A, *is a Banach space under the norm* $\pi_{A,p}(T)$.

PROPOSITION 4. *Let* E, F, G *be normed linear spaces. If* $T \in B(E, F)$ *and* $S : F \to G$ *is an absolutely* $A - p$ *summing operator, then* ST *is an absolutely* $A - p$ *summing operator and* $\pi_{A,p}(ST) \leq \| T \| \pi_{A,p}(S)$.

The proofs of Propositions 3 and 4 follow closely the corresponding results in [1] and [2], and will therefore be omitted.

PROPOSITION 5. *Let* A *satisfy* $\sum_{j=1}^{\infty} a_{nj} \geq \delta > 0$ *for each* $n$. *Then every absolutely* $A - p$ *summing operator is compact.*

*Proof.* Suppose there exists an $\varepsilon > 0$ and an orthonormal sequence $\{e_n\}$ such that $\| Te_n \| > 0$, $n = 1, 2, 3, \cdots$. Then

$$\varepsilon \left(\sum_{n=1}^{k} \delta^{p}\right)^{1/p} \leq \left(\sum_{n=1}^{k} \left(\sum_{j=1}^{\infty} a_{nj}\right)^{p}\right)^{1/p} \leq \left(\sum_{n=1}^{k} \left(\sum_{j=1}^{\infty} a_{nj} \| Te_j \|\right)^{p}\right)^{1/p} \leq$$

$$\leq \pi_{A,p}(T) \sup_{\|a\|\leq 1} \left(\sum_{n=1}^{k} \left(\sum_{j=n}^{\infty} a_{nj} |\langle e_j , a\rangle|\right)^{p}\right)^{1/p} < \infty,$$

a contradiction.

Therefore $\| Te_n \| \to 0$ for every orthonormal sequence $\{e_n\}$, and T is compact.

In [3] the Author established several results for classes of generalized-Hilbert-Schmidt (GHS) operators. The definition used there is of course not unique. Other definitions can be made, which will yield a class of GHS operators satisfying the properties estrablished in [3]. We list here two other such defin tions.

Let A satisfy:

(i) $A \in B(l^2)$,

(II) (ii) $$\sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} a_{nk} \sum_{j=1}^{\infty} a_{nj}\right)^{2} = \infty,$$

(iii) $a_{nk} \geq 0$ for each $n$ and $k$.

Let $T \in B(H_1, H_2)$, $H_i$ separable Hilbert spaces. T is called II GHS if, for every orthonormal basis $\{e_n\}$ of $H_1$, and a fixed A satisfying conditions (II).

$$\sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} a_{nk} \sum_{j=1}^{\infty} a_{nj} \| Te_j \|\right)^{2} < \infty.$$

Let A satisfy:

$$\text{(III)}\qquad \begin{aligned}&\text{(i)}\quad A \in B(l^2)\,,\\ &\text{(ii)}\quad \sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{nj}\right)^2 = \infty\,,\\ &\text{(iii)}\quad a_{nk} \geq 0 \qquad \text{for all } n \text{ and } k\,.\end{aligned}$$

T is called III GHS if, for every orthonormal basis $\{e_n\}$ of $H_1$, and a fixed A satisfying conditions III,

$$\sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{nj}\,\|\mathrm{T}e_j\|\right)^2 < \infty\,.$$

PROPOSITION 6. *Let* E , F *be Hilbert spaces,* T *an absolutely* A — 2 *summing operator. Then* T *is* III GHS.

The proof follows immediately from (1) by setting $x_j = e_j$.

COMMENTS 1). The definition (1) reduces to [2] when A is the identity matrix.

2) Definition (1) can be replaced by other definitions, and the corresponding Propositions 1-6 will remain true. For example, one might replace (1) with

$$\text{(2)}\qquad \left(\sum_{n=1}^{\infty} a_{nk}\left(\sum_{j=1}^{\infty} a_{nj}\,\|\mathrm{T}x_j\|\right)^p\right)^{1/p} \leq \mathrm{C}\sup_{\|a\|\leq 1}\left(\sum_{n=1}^{\infty} a_{nk}\left(\sum_{j=1}^{\infty} a_{nj}\,|\langle x_j\,,a\rangle|\right)^p\right)^{1/p},$$

or with

$$\left(\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty} a_{nk}\sum_{j=1}^{\infty} a_{nj}\,\|\mathrm{T}x_j\|\right)^p\right)^{1/p} \leq \mathrm{C}\sup_{\|a\|\leq 1}\left(\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty} a_{nk}\sum_{j=1}^{\infty} a_{nj}\,|\langle x_j\,,a\rangle|\right)^p\right)^{1/p}.$$

Using [2] gives a generalization of [1].

OPEN QUESTIONS. 1) Does definition (1) possess the monotonicity property? That is, if $1 \leq p < q < \infty$ and T is an absolutely A — $p$ summing operator, must T also be an absolutely A — $q$ summing operator? If so, is $\pi_{A,p}(T) \geq \pi_{A,q}(T)$ ?

2) Is the converse of Proposition 6 true?

## REFERENCES

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[2] A. PIETSCH (1967) – *Absolut p-summierende Abbildungen in normierten Räumen*, « Studia Math. », *28*, 333–353.

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