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Minimal Displacement of Points under Weakly Inward Pseudo-Lipschitzian Mappings, II

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Analisi funzionale. — *Minimal Displacement of Points under Weakly Inward Pseudo-Lipschitzian Mappings*, II. Nota di SIMEON REICH, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — In questa Nota l'Autore continua lo studio di $\inf\{|x - Tx| : x \in C\}$ per trasformazioni T debolmente interne, pseudolipschitziane, definite in un sottoinsieme C limitato e convesso di uno spazio di Banach.

Let E be a real Banach space with norm $|\ |$, and let B (E) denote the family of all nonempty bounded closed convex subsets of E. For each C in B (E) the Chebyshev radius of C is $R(C) = \inf\{\sup\{|x-y| : y \in C\} : x \in C\}$. For z in E we set $d(z, C) = \inf\{|z-y| : y \in C\}$. If $k \geq 0$ we denote by $L(C, k)$ the family of those mappings $T : C \to C$ which satisfy a Lipschitz condition with constant k (that is, $|Tx - Ty| \leq k|x-y|$ for all x and y in C). A mapping $T : C \to E$ is said to be pseudo-lipschitzian with constant k if for all positive r and x, y in C, $|x - y - r(Tx - Ty)| \geq (1 - rk)|x-y|$. It is called weakly inward if $\lim_{h \to 0+} d((1-h)x + hTx, C)/h = 0$ for each $x \in C$. We shall denote the family of continuous $T : C \to E$ which are both weakly inward and pseudo-lipschitzian with constant k by $W(C, k)$. This class of mappings in interesting because T belongs to it is and only if $T - I$ (where I denotes the identity) is a continuous strong generator of a nonlinear sem group of type $k - 1$ on C [4, p. 411].

In a recent paper [2], Goebel studied the quantity $\inf\{|x - Tx| \,|\, x : \in C\}$ for T in $L(C, k)$. He defined, for each E and k, a function $g(E, k) : [0, \infty] \to [0, 1)$ by $g(E, k) = \sup\{\inf\{|x - Tx| : x \in C\}/R(C) : C \in B(E)$ with $R(C) > 0$ and $T \in L(C, k)\}$, and determined some of its properties. In particular, he showed that the right derivative $g'_+(E, 1)$ always exists [2, p. 156]. In order to investigate $\inf\{|x - Tx| : x \in C\}$ for T in $W(C, k)$ we defined [5], for each E and k, $p(E, k) = \sup\{\inf\{|x - Tx| : x \in C\}/R(C) : C \in B(E)$ with $R(C) > 0$ and $T \in W(C, k)\}$. If $k < 1$, then $p(E, k) = 0$ for all E [4, p. 413]. If E is finite-dimensional, then $p(E, k) = 0$ for all k [3, p. 356]. Therefore, we may assume that E is infinite-dimensional and $k \geq 1$. If E is fixed (but arbitrary), we write $p(k)$ (and $g(k)$) instead of $p(E, k)$ (and $g(E, k)$). In [5, Theorem 3] we established the inequality $p(k) \leq (k-1)g'_+(1)$ for all $k \geq 1$. In this supplementary note we show that in fact $p(k)$ equals $(k-1)g'_+(1)$ for all $k \geq 1$.

THEOREM. $p(k) = (k-1)g'_+(1)$ for $k \geq 1$.

(*) Nella seduta del 14 febbraio 1976.

Proof. Let $k \geq 1$ be given. It is sufficient to show that $p(k) \geq \geq (k-1) g'_+(1)$. To this end, let $t > 1$ and $\varepsilon > 0$ be given. There are $C \subset E$ and $T \in L(C, t)$ such that $|Tx - x| \geq (g(t) - \varepsilon) R(C)$ for all x in C. Let $m = (k-1)/(t-1)$. Define $S : C \rightarrow E$ by $S = (1-m) I + mT$. Since $Sx = x + m(Tx - x)$ for each x in C, S is inward [3] and therefore weakly inward. Moreover, S in continuous and pseudo–lipschitzian with constant $1 - m + mt = k$. Therefore S belongs to $W(C, k)$. For x in C, $|x - Sx| = m |x - Tx| \geq m(g(t) - \varepsilon) R(C)$. This means that $p(k) \geq \geq (k-1)(g(t) - \varepsilon)/(t-1)$ for all $\varepsilon > 0$ and $t > 1$. Consequently, $p(k) \geq \geq (k-1) g(t)/(t-1)$ for all $t > 1$, and the results follows.

COROLLARY 1. $p(k) \leq k - 1$ *with equality if and only if* $g(k) = 1 - 1/k$.

COROLLARY 2. $p(k) = g(k)$ *if and only if* $g(k) = 0$.

Similar results can be obtained when $R(C)$ is replaced by $\inf \{\sup \{d(y, F) : y \in C\} : F \subset C \text{ is finite}\}$ (cfr. [1]).

REFERENCES

[1] M. FURI and M. MARTELLI (1974) – *On the minimal displacement of points under α–Lipschitz maps in normed spaces*, « Boll. Un. Mat. Ital. », *9*, 791–799.

[2] K. GOEBEL (1973) – *On the minimal displacement of points under lipschitzian mappings*, « Pacific J. Math. », *45*, 151–163.

[3] B. R. HALPERN and G. M. BERGMAN (1968) – *A fixed point theorem for inward and outward maps*, « Trans. Amer. Math. Soc. », *130*, 353–358.

[4] R. H. MARTIN Jr. (1973) – *Differential equations on closed subsets of a Banach space*, « Trans. Amer. Math. Soc. », *179*, 399–414.

[5] S. REICH – *Minimal displacement of points under weakly inward pseudo-lipschitzian mappings*, « Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Natur. », to appear.