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Analisi matematica. — *A Nullstellensatz for analytic ideals of differentiable functions.* Nota di WILLIAM A. ADKINS e JOHN V. LEAHY, presentata (*) dal Corrisp. A. ANDREOTTI.

RIASSUNTO. — Si dimostra una estensione del teorema degli zeri di Hilbert per un ideale di funzioni indefinitamente differenziabili e generato da funzioni analitiche.

1. INTRODUCTION

In [1] Bochnak proves that a real ideal of C^∞ functions which is finitely generated by analytic functions is equal to the ideal of all C^∞ functions which vanish on the zeros of the ideal. This theorem is proved by showing that if an ideal is real then the regular points of the ideal are M-dense in the sense of Malgrange. Then by the M-density theorem of Malgrange the closure of the ideal is determined by applying the Whitney spectral theorem only at the regular points. The theorem then follows from the reality of the ideal and the fact that any ideal of C^∞ functions which is finitely generated by analytic functions is closed.

In this paper we prove that for any ideal of C^∞ functions which is generated by analytic functions the ideal of C^∞ functions which vanish on the zeros of the ideal is equal to the closure of the real radical of the ideal. This theorem has Bochnak's result as a corollary; however it is proved by a different method.

2. PRELIMINARIES

a) Let A be a commutative ring with identity and $\mathfrak{A}$ an ideal of A. Then $\mathfrak{A}$ is a real ideal if whenever $a_1^2 + \cdots + a_r^2 \in \mathfrak{A}$, $a_i \in A$ if follows that $a_i \in \mathfrak{A}$ for $1 \leq i \leq r$. The real radical of $\mathfrak{A}$, denoted by $\sqrt[\mathbf{R}]{\mathfrak{A}}$, is defined to be the intersection of all the real prime ideals of A which contain $\mathfrak{A}$. It is not difficult to show that $\sqrt[\mathbf{R}]{\mathfrak{A}} = \{ a \in A : \exists k \in N , a_i \in A \text{ for } 1 \leq i \leq s \text{ with } a^{2k} + a_1^2 + \cdots + a_s^2 \in \mathfrak{A} \}$ and that $\mathfrak{A}$ is a real ideal if and only if $\mathfrak{A} = \sqrt[\mathbf{R}]{\mathfrak{A}}$.

If A is a noetherian ring then it can also be shown that $\sqrt[\mathbf{R}]{\mathfrak{A}} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s$ where $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_s\}$ consists of the minimal elements of the set of all real prime ideals of A containing $\mathfrak{A}$.

The paper by Dubois and Efroymson [2] contains complete proofs of all the preceding remarks on real ideals. Also see Galbiati-Tognoli [3].

(*) Nella seduta del 14 febbraio 1976.

b) If Ω is any open subset of $\mathbf{R}^n$ let $\mathscr{E}(\Omega)$ denote the $\mathbf{R}$-algebra of all real valued C^∞ functions on Ω and $\mathscr{A}(\Omega)$ the $\mathbf{R}$-subalgebra of $\mathscr{E}(\Omega)$ whose elements are the real analytic functions on Ω. We give $\mathscr{E}(\Omega)$ the C^∞ topology. If E is any subset of $\mathscr{E}(\Omega)$ and X any subset of Ω let $V(E) = \{x \in \Omega : f(x) = 0 \ \forall f \in E\}$ and $I^E(X) = \{f \in E : f(x) = 0 \ \forall x \in X\}$. Moreover, if $x_0 \in \Omega$ let E_{x_0} (respectively X_{x_0}) be the germ of E (respectively X) at x_0. We also define $V_{x_0}(E_{x_0}) = V(E)_{x_0}$ and $I^{E_{x_0}}(X_{x_0}) = I^E(X)_{x_0}$. Denote the closure of E in $\mathscr{E}(\Omega)$ by $\bar{E}$. Let $\mathscr{F}_{x_0}$ denote the $\mathbf{R}$-algebra of formal power series at x_0 and $T_{x_0} : \mathscr{E}_{x_0} = \mathscr{E}(\Omega)_{x_0} \to \mathscr{F}_{x_0}$ the taylor map. Let $I^{\mathscr{F}_{x_0}}(X_{x_0}) = T_{x_0}(I^{\mathscr{E}_{x_0}}(X_{x_0})) \hookrightarrow \mathscr{F}_{x_0}$

C) PROPOSITION. *If Ω is an open subset of $\mathbf{R}^n$ and $J \hookrightarrow \mathscr{A}(\Omega)$ is an ideal, then* $J \cdot \mathscr{E}(\Omega) \cap \mathscr{A}(\Omega) = J$.

Proof. Suppose that $f = \sum_{i=1}^{r} h_i f_i \in J \cdot \mathscr{E}(\Omega) \cap \mathscr{A}(\Omega)$ where $f_i \in J$ and $h_i \in \mathscr{E}(\Omega)$ for $1 \leq i \leq r$. Let $\mathfrak{c} = \mathfrak{c}(f, f_1, \cdots, f_r)$ be the sheaf of $\mathscr{A}$-ideals generated by $f, f_1, \cdots, f_r$. Consider the sheaf homomorphism $\varphi : \mathscr{A}^r \to \mathfrak{c}$ defined by $\varphi(g_{1x}, \cdots, g_{rx}) = g_{1x} \cdot f_{1x} + \cdots + g_{rx} \cdot f_{rx}$ where $g_{ix} \in \mathscr{A}_x$ for $1 \leq i \leq r$. Note that for $x \in \Omega$

$$f_x = T_x f = \sum_{i=1}^{r} (T_x h_i)(T_x f_i) = \sum_{i=1}^{r} (T_x h_i) \cdot f_{ix}.$$

Since $\mathscr{F}_x$ is faithfully flat over $\mathscr{A}_x$ we have that $f_x = \sum_{i=1}^{r} g_{ix} \cdot f_{ix}$ where $g_{ix} \in \mathscr{A}_x$ for $1 \leq i \leq r$. Thus φ is surjective and there is an exact sequence of sheaves $0 \to \mathscr{K} \to \mathscr{A}^r \to \mathfrak{c} \to 0$ where $\mathscr{K} = \ker \varphi$. Since $\mathfrak{c}$ and $\mathscr{A}^r$ are coherent, $\mathscr{K}$ is coherent. Hence $H^1(\Omega, \mathscr{K}) = 0$ and the map $\Gamma(\Omega, \mathscr{A}^r) \xrightarrow{\varphi} \Gamma(\Omega, \mathfrak{c})$ is surjective. Since $f \in \Gamma(\Omega, \mathfrak{c})$ it follows that $f \in J$. Therefore $J \cdot \mathscr{E}(\Omega) \cap \mathscr{A}(\Omega) \subseteq J$ and since $J \subseteq J \cdot \mathscr{E}(\Omega) \cap \mathscr{A}(\Omega)$ is obvious, we are done.

COROLLARY. *If* $J \cdot \mathscr{E}(\Omega)$ *is a real ideal in* $\mathscr{E}(\Omega)$ *then* J *is a real ideal in* $\mathscr{A}(\Omega)$.

Remark. Unfortunately, the converse to the above corollary is false in general. For example, it can be shown that the ideal $(x_3(x_1^2 + x_2^2) - x_1^3) \cdot \mathscr{A}(\mathbf{R}^3) \hookrightarrow \mathscr{A}(\mathbf{R}^3)$ is a real ideal in $\mathscr{A}(\mathbf{R}^3)$, but $(x_3(x_1^2 + x_2^2) - x_1^3) \cdot \mathscr{E}(\mathbf{R}^3)$ is not a real ideal in $\mathscr{E}(\mathbf{R}^3)$.

3. THE NULLSTELLENSATZ

In [6] Risler proved that if $J_x \hookrightarrow \mathscr{A}_x$ then $I^{\mathscr{A}_x}(V_x(J_x)) = J_x$ if and only if J_x is a real ideal. Since $\mathscr{A}_x$ is a noetherian ring, our previous remarks on the real radical of an ideal in a noetherian ring show that Risler's theorem can be stated as follows.

THEOREM (Risler). *If* $J_x \hookrightarrow \mathscr{A}_x$, *then* $I^{\mathscr{A}_x}(V_x(J_x)) = \sqrt[\mathbf{R}]{J_x}$.

We will also need the following theorem of Malgrange [4].

THEOREM (Malgrange). *If* X_x *is the germ at* $x \in \mathbf{R}^n$ *of a real analytic variety, then*

$$I^{\mathscr{A}_x}(X_x) \cdot \mathscr{F}_x = I^{\mathscr{F}_x}(X_x).$$

THEOREM. *If* Ω *is an open subset of* $\mathbf{R}^n$ *and* $J \hookrightarrow \mathscr{A}(\Omega)$ *an ideal of real analytic functions on* Ω, *then*

$$I^{\mathscr{E}(\Omega)}(V(J \cdot \mathscr{E}(\Omega))) = \overline{\sqrt[\mathbf{R}]{J \cdot \mathscr{E}(\Omega)}} \hookrightarrow \mathscr{E}(\Omega).$$

Proof. Since $I^{\mathscr{E}(\Omega)}(V(J \cdot \mathscr{E}(\Omega)))$ is a closed ideal in $\mathscr{E}(\Omega)$ which contains the ideal $\sqrt[\mathbf{R}]{J \cdot \mathscr{E}(\Omega)}$, we have

$$\overline{\sqrt[\mathbf{R}]{J \cdot \mathscr{E}(\Omega)}} \subseteq I^{\mathscr{E}(\Omega)}(V(J \cdot \mathscr{E}(\Omega))).$$

The reverse inclusion will follow from Whitney's spectral theorem if we show that $T_x(I^{\mathscr{E}(\Omega)}(V(J \cdot \mathscr{E}(\Omega)))) \subseteq T_x(\sqrt[\mathbf{R}]{J \cdot \mathscr{E}(\Omega)})$ for all $x \in V(J \cdot \mathscr{E}(\Omega))$. Let x be any point in the analytic subset $V(J \cdot \mathscr{E}(\Omega)) = V(J)$ of Ω. Then we have

1) $I^{\mathscr{A}_x}(V(J)_x) \cdot \mathscr{F}_x = I^{\mathscr{F}_x}(V(J)_x)$ by Malgrange's theorem, and

2) $\sqrt[\mathbf{R}]{J_x} = I^{\mathscr{A}_x}(V_x(J_x)) = I^{\mathscr{A}_x}(V(J_x))$ by Risler's theorem.

To combine 1) and 2) we need the following.

LEMMA $\sqrt[\mathbf{R}]{J_x} \subseteq (\sqrt[\mathbf{R}]{J \cdot \mathscr{E}(\Omega)})_x$.

Proof. Let $f_x \in \sqrt[\mathbf{R}]{J_x}$. Then there exists $\varphi_{ix} \in \mathscr{A}_x$ for $1 \leq i \leq r$ and $k \in N$ such that $f_x^{2k} + \varphi_{1x}^2 + \cdots + \varphi_{rx}^2 \in J_x$. Let U be an open neighborhood of x in Ω and $f, \varphi_1, \cdots, \varphi_r$ elements of $\mathscr{A}(U)$ which represent $f_x, \varphi_{1x}, \cdots, \varphi_{rx}$ on U. Then there exists $g_1, \cdots, g_s$ in J and $h_1, \cdots, h_s$ in $\mathscr{A}(U)$ (shrink U if necessary) such that $f^{2k} + \varphi_1^2 + \cdots + \varphi_r^2 = \sum_{i=1}^{s} h_i g_i$ on U. Let $\psi \in \mathscr{E}(\mathbf{R}^n)$ such that $\psi(x) = 1$ in a neighborhood of x and $\operatorname{supp} \psi \subseteq U$. Then

$$(f \cdot \psi)^{2k} + (\varphi_1 \cdot \psi^k)^2 + \cdots + (\varphi_r \cdot \psi^k)^2 = \sum_{i=1}^{s} (\psi^{2k} \cdot h_i) \cdot g_i$$

on Ω and hence $\psi \cdot f \in \sqrt[\mathbf{R}]{J \cdot \mathscr{E}(\Omega)}$. Since $\psi = 1$ near x it follows that $f_x \in (\sqrt[\mathbf{R}]{J \cdot \mathscr{E}(\Omega)})_x$ and the lemma is proved.

We now have

3) $I^{\mathscr{F}_x}(V(J \cdot \mathscr{E}(\Omega))_x) = \sqrt[\mathbf{R}]{J_x} \cdot \mathscr{F}_x \subseteq (\sqrt[\mathbf{R}]{J \cdot \mathscr{E}(\Omega)})_x \cdot \mathscr{F}_x = T_x(\sqrt[\mathbf{R}]{J \cdot \mathscr{E}(\Omega)})$

for all $x \in V(J \cdot \mathscr{E}(\Omega))$. Since $T_x(I^{\mathscr{E}(\Omega)}(V(J \cdot \mathscr{E}(\Omega))))$ is clearly contained in $I^{\mathscr{F}_x}(V(J \cdot \mathscr{E}(\Omega))_x)$, we have

$$T_x(I^{\mathscr{E}(\Omega)}(V(J \cdot \mathscr{E}(\Omega)))) \subseteq T_x(\sqrt[\mathbf{R}]{J \cdot \mathscr{E}(\Omega)})$$

for all $x \in V(J \cdot \mathscr{E}(\Omega))$ and the proof of the theorem is complete.

COROLLARY. *Let* $\mathrm{H} \hookrightarrow \mathscr{E}(\Omega)$ *be an ideal generated by finitely many real analytic functions. Then* $\mathrm{I}^{\mathscr{E}(\Omega)}(\mathrm{V}(\mathrm{H})) = \mathrm{H}$ *if and only if* H *is a real ideal in* $\mathscr{E}(\Omega)$.

Proof. If $\mathrm{I}^{\mathscr{E}(\Omega)}(\mathrm{V}(\mathrm{H})) = \mathrm{H}$ then H is clearly real. Now suppose that H is real. Since H is generated by finitely many real analytic functions H is closed [4]. Hence

$$\mathrm{I}^{\mathscr{E}(\Omega)}(\mathrm{V}(\mathrm{H})) = \overline{\sqrt[\mathbf{R}]{\mathrm{H}}} = \overline{\mathrm{H}} = \mathrm{H}.$$

4. EXAMPLES

There is an ideal $\mathrm{J} \hookrightarrow \mathscr{A}(\mathbf{R}^2)$ such that

$$\mathrm{J}\cdot\mathscr{E}(\mathbf{R}^2) \subsetneqq \sqrt{\mathrm{J}\cdot\mathscr{E}(\mathbf{R}^2)} \subsetneqq \sqrt[\mathbf{R}]{\mathrm{J}\cdot\mathscr{E}(\mathbf{R}^2)} \subsetneqq \overline{\sqrt[\mathbf{R}]{\mathrm{J}\cdot\mathscr{E}(\mathbf{R}^2)}} = \mathrm{I}^{\mathscr{E}(\mathbf{R}^2)}(\mathrm{V}(\mathrm{J}\cdot\mathscr{E}(\mathbf{R}^2))) .$$

Example 1. If $\mathrm{E}_p(x) = (1 - x)\exp\left(\frac{x}{1} + \frac{x^2}{2} + \cdots + \frac{x^p}{p}\right)$ for $x \in \mathbf{R}$, let $f(x) = \prod_{n=1}^{\infty}\left(\mathrm{E}_{p_n}\left(\frac{x}{n}\right)\right)^n$ where p_n are chosen large enough so that $f(x)$ and $f_1(x) = x \cdot \prod_{n=1}^{\infty} \mathrm{E}_{p_n}\left(\frac{x}{n}\right)$ converge uniformly on compact sets. Define $g \in \mathscr{A}(\mathbf{R}^2)$ by $g(x, y) = (x^2 + y^2) f(x)$ and let $\mathrm{J} = (g)\cdot\mathscr{A}(\mathbf{R}^2) \hookrightarrow \mathscr{A}(\mathbf{R}^2)$. Clearly $\mathrm{J}\cdot\mathscr{E}(\mathbf{R}^2) \subsetneqq \sqrt{\mathrm{J}\cdot\mathscr{E}(\mathbf{R}^2)}$. Since $f_2(x, y) = xf(x) \in \sqrt[\mathbf{R}]{\mathrm{J}\cdot\mathscr{E}(\mathbf{R}^2)}$ and $f_2 \notin \sqrt{\mathrm{J}\cdot\mathscr{E}(\mathbf{R})^2}$ we have $\sqrt{\mathrm{J}\cdot\mathscr{E}(\mathbf{R}^2)} \subsetneqq \sqrt[\mathbf{R}]{\mathrm{J}\cdot\mathscr{E}(\mathbf{R}^2)}$. The Whitney spectral theorem gives $\overline{\sqrt[\mathbf{R}]{\mathrm{J}\cdot\mathscr{E}(\mathbf{R}^2)}} = (y, f_1)\cdot\mathscr{E}(\mathbf{R}^2)$ and it is easy to see that $f_1 \notin \sqrt[\mathbf{R}]{\mathrm{J}\cdot\mathscr{E}(\mathbf{R}^2)}$. Hence $\sqrt[\mathbf{R}]{\mathrm{J}\cdot\mathscr{E}(\mathbf{R}^2)} \subsetneqq \overline{\sqrt[\mathbf{R}]{\mathrm{J}\cdot\mathscr{E}(\mathbf{R}^2)}}$.

The following example shows that the theorem is not true for ideals which are not generated by real analytic functions.

Example 2. Let $\mathscr{M}_0^\infty \hookrightarrow \mathscr{E}(\mathbf{R}^n)$ be the ideal of all C^∞ functions on $\mathbf{R}^n$ which are flat at $0 \in \mathbf{R}^n$. The Whitney spectral theorem shows that $\mathscr{M}_0^\infty$ is closed, and it follows easily from Leibnitz' rule that $\mathscr{M}_0^\infty$ is real. Hence $\mathscr{M}_0^\infty = \overline{\sqrt[\mathbf{R}]{\mathscr{M}_0^\infty}}$ but

$$\mathrm{I}^{\mathscr{E}(\mathbf{R}^n)}(\mathrm{V}(\mathscr{M}_0^\infty)) = \mathrm{I}^{\mathscr{E}(\mathbf{R}^n)}(\{0\}) = \mathscr{M}_0 \supsetneqq \mathscr{M}_0^\infty .$$

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