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A further account of roto-translations and the use of the method of conditioned observations

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Fotogrammetria. — *A further account of roto-translations and the use of the method of conditioned observations.* Nota di FERDINANDO SANSÒ, presentata (*) dal Socio L. SOLAINI.

RIASSUNTO. — La rototraslazione con variazione delle dimensioni di un modello fotogrammetrico è di fondamentale importanza nell'orientamento assoluto e nella triangolazione aerea. Il problema presenta qualche difficoltà di calcolo in presenza di elementi esuberanti. La presente Nota lo risolve usando quaternioni ed il metodo delle osservazioni condizionate con parametri incogniti.

It has been demonstrated in several ways in recent years that the problem of the calculation of a roto-translation and scale factor between two systems of ground coordinates, starting from a certain number of points, has an exact solution in the sense that the least squares method applied to the error equations leads to equations solvable by means of appropriate algebraic devices.

In particular, it has been proved [1] that, by using an appropriate representation of the rotations by means of quaternions, the problem is reduced to the search of the minimum eigenvalue and of the corresponding eigenvector of a certain symmetric and negative matrix. On the other hand, the application of the least squares method to the residuals of the observation equations gave rise to many objections; in fact, it seems more correct, on the contrary, to apply the same method to the corrections of the quantities measured using the equations wich express physical and geometrical relations as condition-equations with unknown parameters.

The purpose of these pages is to demonstrate that the problem, stated also in these terms, can be reduced to the calculation of the minimum eigenvalue and of the corresponding eigenvector of a given matrix.

1. REPRESENTATION BY QUATERNIONS AND STATEMENT OF THE METHOD LEAST SQUARES

We remind the reader that in representing three-dimensional vectors as imaginary quaternions $r = i\boldsymbol{r}$, the rotations can correspondingly be represented by quaternions $q = q_0 + i\boldsymbol{q}$ with unitary modulus

$$|q| = \sqrt{q_0^2 + |\boldsymbol{q}|^2} = 1 .$$

(*) Nella seduta del 14 febbraio 1976.

(1) FERNANDO SANSÒ (1973) - *An exact solution of the roto-translation problem*, «Photogrammetria», *29* (1973), 203-216.

More precisely, if B is a rotation matrix, and if we let q correspond to B so that

$$(1.1)\qquad \mathrm{B} = \begin{bmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & -2q_0q_3+2q_2q_1 & 2q_0q_2+2q_3q_1 \\ 2q_0q_3+2q_2q_1 & q_0^2-q_1^2+q_2^2-q_3^2 & -2q_0q_1+2q_3q_2 \\ -2q_0q_2+2q_1q_3 & 2q_0q_1+2q_2q_3 & q_0^2-q_1^2-q_2^2+q_3^2 \end{bmatrix}$$

$$q_0^2+q_1^2+q_2^2+q_3^2 = 1$$

then, to the three-dimensional rotation

$$\boldsymbol{s} = \mathrm{B}\boldsymbol{r}$$

there corresponds the rotation in the space of the imaginary quaternions

$$s = i\boldsymbol{s} \quad , \quad r = i\boldsymbol{r}; \quad \text{we have } s = qr\bar{q} \quad (\bar{q} = q_0 - i\boldsymbol{q}).$$

Using this representation, a roto-translation, which in vector form is written as

$$\boldsymbol{s} = \boldsymbol{b} + \rho\mathrm{B}\boldsymbol{r}$$

(b, translation; B, rotation; ρ, scale factor)

becomes

$$(1.2)\qquad s = b + \rho qr\bar{q}.$$

Formula (1.2) is the generating equation of the observation equations system, once some points r_i are known in the system of reference (R) and the corresponding points s_i in the system of reference (S).

The solution of the problem stated above is reported in the above mentioned paper and has led to the solution for the eigenvalues:

$$\sum s_i q r_i - \lambda q = 0 \qquad (\lambda = \lambda_{\min}).$$

If, on the contrary, we use formula (1.2) as a condition equation, we have to introduce as unknowns the corrections v_i to be given to the measured points r_i (points in the model system):

$$(1.3)\qquad r_i^* = r_i + v_i$$

and (1.2) becomes:

$$(1.4)\qquad s_i - b - \rho q r_i \bar{q} - \rho q v_i \bar{q} = 0.$$

Application of the least squares method thus leads us to seek the minimum of

$$\sum |v_i|^2 = \sum \bar{v}_i v_i ,$$

with conditions (1.4) and $|q|^2 = 1$.

For convenience, it is worth introducing the new unknowns

$$(1.5)\qquad u_i = qv_i\bar{q};$$

since, obviously,

$$|u_i|^2 = |v_i|^2 ,$$

the problem consists in finding

$$\min \sum_i |u_i|^2 \tag{1.6}$$

with the conditions

$$s_i - b - \rho q r_i \bar{q} - \rho u_i = 0 \tag{1.7}$$

$$|q|^2 = \bar{q} q = 1 . \tag{1.8}$$

2. Solution of the problem

The problem, stated in the previous paragraph, can be solved by introducing Lagrange multipliers for conditions (1.7) and (1.8): in particular, since (1.7) are equations whose terms are imaginary quaternions, it is necessary to introduce for each of them a $\overline{\gamma}_i$ multiplier which is also an imaginary quaternion, while, as (1.8) is a scalar equation, we can introduce for it, as Lagrange multiplier, a scalar α.

Thus the problem becomes the search for the free minimum of the function:

$$\Phi = \tfrac{1}{2} \sum \bar{u}_i u_i + \operatorname{Re} \sum \overline{\gamma}_i (s_i - b - \rho q r_i \bar{q}) + \tfrac{1}{2} \alpha (\bar{q} q - 1) \tag{2.1}$$

with respect to the unknowns u_i , b , ρ , q.

Taking into account some properties of the quaternions and in particular

$$\operatorname{Re}(pq) = \operatorname{Re}(qp) = \operatorname{Re}(\bar{q}\bar{p}) = \operatorname{Re}(\bar{p}\bar{q}) , \tag{2.2}$$

and that, for the imaginary quaternions

$$\bar{r} = -r , \tag{2.3}$$

from the condition $d\Phi = 0$ with respect to each unknown, we obtain the equations

$$u_i - \rho \gamma_i = 0 \tag{2.4}$$

$$-\Sigma \overline{\gamma}_i = 0 \tag{2.5}$$

$$\operatorname{Re} \sum \overline{\gamma}_i (q r_i \bar{q} + u_i) = 0 \tag{2.6}$$

$$2 \rho \sum \gamma_i q r_i + \alpha q = 0 . \tag{2.7}$$

Besides these equations, conditions (1.7) and (1.8) also have to be satisfied.

From (2.4) and (2.5) we get:

$$\sum \gamma_i = 0 \qquad \text{from which} \quad \sum u_i = 0$$

that is

$$\sum s_i - nb - \rho q \left(\sum r_i\right) \bar{q} = 0 . \tag{2.8}$$

If the origins of the systems (R) and (S) lie in the centres of gravity of points r_i and s_i, then

$$\sum s_i = \sum r_i = 0$$

and therefore, in addition, as is obvious,

$$b = 0 . \tag{2.9}$$

Taking into account (2.9), (1.7) becomes

$$s_i - \rho q r_i \bar{q} - \rho u_i = 0 . \tag{2.10}$$

From (2.6) and (2.10) we have

$$\mathrm{Re} \sum \bar{\gamma}_i s_i = 0 ;$$

subsequently, recalling (2.4)

$$\mathrm{Re} \sum \bar{u}_i s_i = 0$$

$$\sum \bar{s}_i s_i - \rho \, \mathrm{Re} \sum q \bar{r}_i \bar{q} s_i = 0$$

that is

$$\mathrm{Re} \sum q r_i \bar{q} s_i = -\frac{1}{\rho} \sum | s_i |^2 . \tag{2.11}$$

From (2.7), keeping in mind (2.4) and (2.10) we have:

$$\frac{2}{\rho} \sum s_i q r_i - 2 \sum q r_i \bar{q} q r_i + \alpha q = 0 ,$$

and therefore, remembering that $\bar{q} q = 1$ and $r_i^2 = - | r_i |^2$,

$$\frac{2}{\rho} \sum s_i q r_i + \left(2 \sum | r_i |^2 + \alpha\right) q = 0 . \tag{2.12}$$

Now by multiplying on the right (2.12) by $\bar{q}$ and taking the part of it which is real, one obtains

$$\frac{2}{\rho} \mathrm{Re} \left(\sum s_i q r_i \bar{q}\right) + 2 \sum | r_i |^2 + \alpha = 0 . \tag{2.13}$$

Comparing (2.13) with (2.11) and remembering the property (2.2), a relation is found which allows elimination of the multiplier α

$$2 \sum | r_i |^2 + \alpha = \frac{2}{\rho^2} \sum | s_i |^2 \tag{2.14}$$

which, substituted in (2.12), finally gives

$$\sum s_i q r_i + \left(\frac{1}{\rho} \sum |s_i|^2\right) q = 0 . \tag{2.15}$$

This, as can be seen, has led again to an eigenvalue equation identical to the one considered in the previously cited paper: an expansion of (2.15) in its four components, leads to the search of the eigenvalues of the system

$$Aq - \lambda q = 0$$

where

$$A = -\begin{bmatrix} \frac{1}{n}\sum_{ik} r_i^k s_i^k & \frac{1}{n}\sum_i (r_i^2 s_i^3 - r_i^3 s_i^2) & \frac{1}{n}\sum_i (r_i^3 s_i^1 - r_i^1 s_i^3) & \frac{1}{n}\sum_i (r_i^1 s_i^2 - r_i^2 s_i^1) \\ & -\frac{1}{n}\sum_{ik} r_i^k s_i^k + \frac{2}{n}\sum_i s_i^1 r_i^1 & \frac{1}{n}\sum_i (r_i^2 s_i^1 + r_i^1 s_i^2) & \frac{1}{n}\sum_i (r_i^3 s_i^1 + r_i^1 s_i^3) \\ & & -\frac{1}{n}\sum_{ik} r_i^k s_i^k + \frac{2}{n}\sum_i r_i^2 s_i^2 & \frac{1}{n}\sum_i (r_i^3 s_i^2 + r_i^2 s_i^3) \\ & & & -\frac{1}{n}\sum_{ik} r_i^k s_i^k + \frac{2}{n}\sum_i r_i^3 s_i^3 \end{bmatrix}$$

$A^T = A$; $r_i^k , s_i^k \ (k = 1 , 2 , 3)$, components of r_i , s_i .

It is easy to prove that the required eigenvalue of such a matrix is precisely the minimum one, which geometrically corresponds to:

$$\lambda_{\min} = - \operatorname{Re} \sum \bar{s}_i q r_i \bar{q} = - \sum \boldsymbol{s}_i \cdot B \boldsymbol{r}_i . \tag{2.16}$$

Comparing the eigenvalue equation

$$\sum s_i q r_i - \lambda q = 0 \qquad \text{that is} \quad Aq - \lambda q = 0$$

with (2.15), it follows that the scale factor is given by

$$\rho = - \frac{\sum |s_i|^2}{\lambda_{\min}} . \tag{2.17}$$

It is possible to demonstrate that the value found in this way differs from the value found by applying the method of indirect observations, by quantities of the order of the ratio $\dfrac{\sum |v_i|^2}{\sum |r_r|^2}$.

Let us now take into consideration two possible generalizations of the result obtained.

3. SOLUTION WHEN THE MEASURED POINTS HAVE DIFFERENT WEIGHTS

If the measured vectors r_i, and correspondingly their residuals u_i, have different weights p_i, the function whose minimum has to be found is:

$$\Phi = \tfrac{1}{2} \sum p_i \bar{u}_i u_i + \mathrm{Re} \sum \bar{\gamma}_i (s_i - b - \rho q r_i \bar{q} - \rho u_i) + \tfrac{1}{2} \alpha (\bar{q} q - 1) , \tag{3.1}$$

together with the usual conditions (1.7) and (1.8).

Let us first demonstrate that, when the origins of the system (R) and (S) satisfy the conditions

$$\sum p_i r_i = 0 \quad , \quad \sum p_i s_i = 0 , \tag{3.2}$$

the translation vector b is zero.

In fact, by putting equal to zero the derivative of Φ with respect to u_i we obtain

$$p_i u_i - \rho \gamma_i = 0, \tag{3.3}$$

whereas by putting equal to zero the derivative of Φ with respect to b we find.

$$\sum \gamma_i = 0 \tag{3.4}$$

Thus, from (3.3) and (3.4) we obtain

$$\sum p_i u_i = 0 .$$

i.e.

$$\sum p_i (s_i - b - \rho q r_i \bar{q}) = 0$$

which, given the conditions (3.2), yields precisely

$$b = 0 . \tag{3.5}$$

Having established this result it is worth transforming the variables, letting

$$\sqrt{p_i}\, r_i = R_i \quad ; \quad \sqrt{p_i}\, s_i = S_i \quad ; \quad \sqrt{p_i}\, u_i = U_i . \tag{3.6}$$

Using these new variables the problem now consists in finding the minimum of

$$\Phi = \tfrac{1}{2} \sum \bar{U}_i U_i + \mathrm{Re} \sum \bar{\gamma}_i (s_i - \rho q R_i \bar{q} - \rho U_i) + \tfrac{1}{2} \alpha (\bar{q} q - 1) \tag{3.7}$$

with the conditions

$$S_i - \rho q R_i \bar{q} - \rho U_i = 0 , \qquad \bar{q} q - 1 = 0. \tag{3.8}$$

As can be seen the problem is formally identical to that of (§ 2) and will have, therefore, the same solution

$$\sum S_i q R_i - \lambda q = 0 \; (\lambda = \lambda_{\min}) \; ; \qquad \rho = - \frac{\sum |S_i|^2}{\lambda_{\min}} \tag{3.9}$$

that is, in terms of the initial variables,

$$\sum p_i s_i q r_i - \lambda q = 0 \, (\lambda = \lambda_{\min}) \; ; \quad \rho = -\frac{\sum p_i |s_i|^2}{\lambda_{\min}} . \tag{3.10}$$

We point out that when not only the different points, but also the individual coordinates of the various points have different weights, then the problem becomes much more complex and it is no longer possible to provide such a compact formulation.

4. Solution of the problem when points of both systems are affected by errors

Let us now suppose that also the points s_i of the terrain are affected by non-negligible errors; thus requiring the introduction of corrections w_i for the measured vectors s_i as well, so that the corrected vectors

$$\overset{*}{s_i} = s_i + w_i \quad ; \quad \overset{*}{r_i} = r_i + v_i \tag{4.1}$$

satisfy the conditions

$$\overset{*}{s_i} - b - \rho q \overset{*}{r_i} \bar{q} = 0 . \tag{4.2}$$

Such conditions, introducing as in the previous paragraphs the "rotated" residual $u_i = q v_i \bar{q}$, can be expressed as

$$s_i + w_i - b - \rho q r_i \bar{q} - \rho u_i = 0 . \tag{4.3}$$

Let us suppose that the vectors u_i and the vectors w_i have the same mean square error: moreover let k be the ratio between the wieghts of the vectors w_i and u_i, that is let

$$k \mathrm{M} (|w_i|^2) = \mathrm{M} (|u_i|^2). \tag{4.4}$$

The application of the method of least squares leads us to look for the minimum of the function

$$k \sum \bar{w}_i w_i + \sum \bar{u}_i u_i . \tag{4.5}$$

Introducing the Lagrange multipliers one has to seek the minimum of

$$\Phi = \frac{k}{2} \sum \bar{w}_i w_i + \frac{1}{2} \sum \bar{u}_i u_i + \tag{4.6}$$

$$+ \operatorname{Re} \bar{\gamma}_i (s_i + w_i - b - \rho q r_i \bar{q} - \rho u_i) + \frac{1}{2} \alpha (\bar{q} q - 1)$$

together with the conditions of (4.3), in addition to the usual condition $\bar{q} q = 1$.

The normal equations obtained from (4.6) are

$$u_i - \rho\gamma_i = 0 \tag{4.7}$$

$$kw_i + \gamma_i = 0 \tag{4.8}$$

$$\sum \bar{\gamma}_i = 0 \tag{4.9}$$

$$\mathrm{Re} \sum \bar{\gamma}_i (q r_i \bar{q} + u_i) = 0 \tag{4.10}$$

$$2\rho \sum \gamma_i q r_i + \alpha q = 0\,. \tag{4.11}$$

From (4.3), (4.7), (4.8), (4.9) it is easy to obtain

$$b = 0 \tag{4.12}$$

provided that the origins of (R) and (S) are situated in the centres of gravity of points r_i and s_i.

From this result and (4.3), (4.7) and (4.8) it is found

$$\rho\gamma_i = u_i = -\rho k w_i = c\,(s_i - \rho q r_i \bar{q}) \qquad \text{with} \quad c = \frac{\rho k}{\rho^2 k + 1}\,. \tag{4.13}$$

Using (4.13) in (4.10) and (4.11) the parameter α can be eliminated and the following eigenvalue equation is obtained

$$\sum s_i q r_i + \frac{1}{1 - 2\rho c}\left[-c \sum |s_i|^2 + \rho\,(1 - \rho c) \sum |r_i|^2\right] q = 0\,. \tag{4.14}$$

As can be seen such an equation always leads to the search of the minimum eigenvalue of the same matrix A, considered previously, and, in fact, q is always determined by the same condition.

On the contrary, the expression of the scale factor varies and in this case is determined by solving the equation derived from (4.13) and (4.14).

$$\frac{\rho k}{1 - \rho^2 k}\left(\sum |s_i|^2 - \frac{1}{k} \sum |r_i|^2\right) = \lambda_{\min}\,. \tag{4.15}$$

As can be seen this a second order equation in the unknown ρ and must be analysed to see if it effectively admits solutions and, in the affirmative, which one has to be chosen.

Recalling (2.16), it is easy to show that equation (4.15) admits two roots ρ_1 and ρ_2, namely

$$\rho_1 < 0 < \rho_2\,; \tag{4.16}$$

the only acceptable solution is then ρ_2.

As a check of this result let us look at the behaviour of ρ_2 when $k \to \infty$: physically speaking, such a condition corresponds to a variance of the residuals w_i, tending to zero that is to the model in which points s_i are without error. Thus we would expect that ρ_2 tends to the value of the scale factor previously found in (2.17).

From (4.15) we derive directly, taking the limit for $k \to \infty$

$$\frac{1}{\rho} \sum |s_i|^2 = -\lambda_{\min} \tag{4.18}$$

which clearly coincides with (2.17).

Conclusions

The method of representing roto-translations by quaternions is clearly a very flexible, easy and geometrically meaningful device.

The solution by means of condition equations with unknown parameters leads to values of rotations and translations identical to those of the classical solution derived by the indirect observation method; only the scale factor undergoes a slight variation.