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Closed geodesics on Finsler manifolds

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Geometria differenziale. — *Closed geodesics on Finsler manifolds* (*).
Nota di FRANCESCO MERCURI (**), presentata (***) dal Socio B. SEGRE.

RIASSUNTO. — Si descrivono alcuni teoremi sull'esistenza di geodetiche chiuse per varietà di Finsler compatte e semplicemente connesse. Le dimostrazioni appariranno in altri lavori.

1. INTRODUCTION

The existence of closed geodesics on a riemannian manifold has been, since Poincaré, a field of very active research in differential geometry. The most general theorem, up to now, reads as follows (see [14]).

Let M be a smooth simply connected manifold of dimension $n \geq 2$. Then:

a) If the cohomology ring of M is not generated by only one element any riemannian metric on M admits infinitely many closed geodesics;

b) There is an open and dense set of riemannian metrics each admitting infinitely many closed geodesics;

c) Any riemannian metric on M admits at least three closed geodesics. Since there are no known examples of compact riemannian manifolds with only finitely many closed geodesics, and the ones left over by *a*) are the " simple ones ", a natural conjecture is that there are always infinitely many. At the Vancouver congress in 1974 Anosov, in his communication, mentioned an example of Finsler metric on S^2 with only two closed geodesics.

It seems interesting therefore to develop, on the lines of the riemannian case, the analogous theory for Finsler metrics. In this paper we will describe the " arithmetic theory " for closed geodesics on Finsler manifolds, and, in this context, the reason for Anosov's example is the non symmetry of the metric. However the arithmetic theory, although very suggestive, does not give a completely satisfactory answer to the problem since arithmetically distinct closed geodesics are not necessarly geometrically distinct.

I wish to thank prof. W. Klingenberg for having suggested the problem and for very helpful conversations, and my advisor at the University of Chicago, prof. R. K. Lashof for his constant encouragement and friendship; most of section 5. comes from his ideas.

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2. Finsler Manifolds

We begin by setting some notations. If M is a smooth manifold we will denote by TM the tangent bundle and by $s_0(\mathrm{TM})$ the zero section. If E_1 and E_2 are smooth vector bundles over a smooth manifold and $f: E_1 \to E_2$ is a smooth fibrewise map, we denote by $d_F f$ the "fibre derivative" of f, i.e. $d_F f(e) = d(f|_{E_{\pi_1(e)}})(e)$, where π_1 is the projection of E_1. Naturally $\mathbf{R}$ will denote the real line, $\mathbf{R}^+$ the non negative real line, $\mathbf{Q}$ the rationals and $\mathbf{Z}$ the integers.

Definition. A Finsler manifold (M , F) is a pair consisting of a smooth manifold M together with a continuous function $F: \mathrm{TM} \to \mathbf{R}^+$ such that:

1) F is C^∞ on $\mathrm{TM}\text{-}s_0(\mathrm{TM})$

2) $F(X) = 0$ if and only if $X \in s_0(\mathrm{TM})$

3) $F(tX) = tF(X)$ for $t \in \mathbf{R}^+$

4) $d_F(F^2): \mathrm{TM}\text{-}s_0(\mathrm{TM}) \to T^*M$ is a non degenerage quadratic form (and therefore positive definite).

The function $E = F^2$ is called the energy of the Finsler metric and it is a C^1 function (and naturally C^∞ outside $s_0(\mathrm{TM})$). The lack of smoothness on $s_0(\mathrm{TM})$ is peculiar of Finsler manifolds and in fact E is C^2 on all TM if and only if F is the norm of a riemannian metric. On the other hand Finsler metrics which are not riemannian "occur in nature" (see, for example, [23]).

Consider the standard symplectic 2-form w_0 on T^*M and the pull-back $w_E = (d_F E)^* w_0$. w_E defines a sympelctic stucture on $\mathrm{TM}\text{-}s_0(\mathrm{TM})$ and a vector field W_E, called the Euler field, by:

$$w_E(W_E, V) = \tfrac{1}{2} dE(V) \qquad \text{for all} \quad V \in T(\mathrm{TM}\text{-}s_0(\mathrm{TM})). \tag{2.1}$$

W_E is then a C^∞ vector field outside the zero section and extends to a C^1 vector field on all TM vanishing on $s_0(\mathrm{TM})$. Moreover it is a second order differential equation.

The geodesics of (M , F) *are defined as those* C^2 *curves whose tangent fields are integral lines of* W_E; it is easily seen that actually geodesics are C^∞ curves and the energy is constant along their tangent fields.

If $(x_1, \cdots, x_n)$ are a system of local coordinates in M and $(x_1, \cdots, x_n, \dot{x}_1, \cdots, \dot{x}_n)$ are the local coordinates in TM associated to the frame $(\partial/\partial x_1, \cdots, \cdots, \partial/\partial x_n)$ the equation 2.1. takes the familiar form:

$$\frac{d}{dt}(\partial E/\partial \dot{x}_i)(x(t), \dot{x}(t)) - \partial E/\partial x_i(x(t), \dot{x}(t)) = 0 . \tag{2.2}$$

For a curve $c: [a, b] \to M$ we define, as in the riemannian case: *the* F*-length* $\tilde{F}$, *and the* E*-length*, $\tilde{E}$, *by*:

$$\tilde{F}(c) = \int_a^b F(\dot{c}(t))\, dt \qquad \tilde{E}(c) = \int_a^b E(\dot{c}(t))\, dt$$

A small modification of the riemannian case argument gives the following:

THEOREM A. (*a*) *The Euler equations* 2.2 *have a unique solution with given initial conditions and this solution is smooth* (C^∞).

(b) *For* $p \in M$ *there is a neigborhood* U_p *of* p *and* $\varepsilon > 0$ *such that any two points* (*in a given order*) *in* U_p *are joined by a unique geodesic of length* $< \varepsilon$ *and this geodesic depends in a* C^∞ *fashion on the two points* (*as long as they are distinct*).

(c) *Geodesics minimize "locally"* $\tilde{E}$ *and* $\tilde{F}$; *conversely any curve that minimizes* $\tilde{E}$ *is a geodesic and any curve that minimized* $\tilde{F}$ *is a geodesic, up to a possible reparametrization.*

3. THE MANIFOLD OF CLOSED CURVES AND THE ENERGY INTEGRAL

Suppose, for simplicity, M embedded in $\mathbf{R}^m$ and consider the Sobolev space $H^1(S, \mathbf{R}^m)$ where $S = \mathbf{R}/\mathbf{Z}$ is the circle parametrized between 0 and 1. The subset $\Lambda M = \{c \in H^1(S, \mathbf{R}^m) : c(S) \subseteq M\}$ has a riemannian manifold structure, moddelled on a Hilbert space, induced by the structure of M (see, for example, [7], [14], [20]).

If $F : TM \to \mathbf{R}^+$ is a Finsler metric on M the energy E induces a map, the E-length o *energy integral*, $\tilde{E} : \Lambda M \to \mathbf{R}$ by:

$$\tilde{E}(c) = \int_S E(\dot{c}(t))\, dt$$

THEOREM B: $\tilde{E}$ *is* C^{2-} (*i.e. it is* C^1 *and its differential is locally lipschitzian*).

Writing $d\tilde{E}$ explicitly it is easy to recognize that any closed geodesic is a critical point for $\tilde{E}$. The converse is also true:

THEOREM C: $c \in M$ *is a critical point for* $\tilde{E}$ *if and only if it is a closed geodesic.*

As mentioned above, the structure of M induces a riemannian structure on ΛM and we will denote by $(,)$ and $\| \cdot \|$ the relative scalar product and norm. Consider then, on ΛM, the vector field $\xi = -\operatorname{grad} \tilde{E}$ defined by

$$(\xi, \eta) = -d\tilde{E}(\eta) \qquad \text{for all} \quad \eta \in T\Lambda M$$

ξ is a C^{1-} vector field and therefore there is a unique integral curve of ξ thru a given point.

The standard technique in Morse–Liustenik-Schnirelman theory for locating the critical points of a function is to "to go down" along the integral lines of the gradient field. This is usually done, in the finite dimentional case, under a properness assumption for the function. The condition that plays the analogous role in the infinite dimensional case is the following (see [14], [18], [19]):

Condition (C). Let H be a riemannian (Hilbert) manifold and $f: \mathrm{H} \to \mathbf{R}$ a C^1 function; f is said to satisfy condition (C) if for any subset $\mathrm{S} \subseteq \mathrm{H}$ on which f is bounded but $\| df \|$ is not bounded away from zero, the closure $\bar{\mathrm{S}}$ of S contains a critical point for f.

The following clearly implies condition (C) for the energy integral:

THEOREM D. *Let* $\{c_n\} \subseteq \Lambda\mathrm{M}$ *be a sequence such that*:

1. $\mathrm{E}(c_n) \leq k_0$ *for some constant* k_0
2. $\| \xi (c_n) \| \to 0$

then $\{c_n\}$ *has a convergent subsequence.*

4. THE CRITICAL POINTS THEORY FOR THE ENERGY INTEGRALS

On $\Lambda\mathrm{M}$, ξ defines a semigroup of $\tilde{\mathrm{E}}$-decreasing transformations $\varphi_t : \Lambda\mathrm{M} \to \Lambda\mathrm{M}$ as follows: Let $c \in \Lambda\mathrm{M}$ and $\psi_c(t)$ be the maximal integral curve of ξ with $\psi_c(0) = c$; then $\psi_c(t)$ is defined for all $t \geq 0$ (as a consequence of condition (C)) and we set $\varphi_t(c) = \psi_c(t)$.

Given a compact set $\mathrm{A} \subseteq \Lambda\mathrm{M}$, $\mathrm{A} \neq \varphi$, we define:

$$\operatorname{minimax}(\mathrm{A}) = \lim_{t \to \infty} \max (\mathrm{E}|_{\varphi_t(\mathrm{A})})$$

The " minimax theorem " gives then (see [19]):

THEOREM E. *Minimax* (A) *is a critical value for* $\tilde{\mathrm{E}}$.

Naturally if we start with different subsets $\mathrm{A}, \mathrm{B} \subseteq \Lambda\mathrm{M}$ the relative minimaxes could coincide and, a priori, could give the same critical point. We will define now a particular class of compact subsets for which the eventual coincidence of critical values (minimaxes) gives the existence of many critical points with that $\tilde{\mathrm{E}}$-value. We fix a field of coefficients for homology and cohomology.

First of all we want to avoid the trivial geodesics, i.e. $\tilde{\mathrm{E}}^{-1}(0)$, and for $a \geq 0$ we set $\Lambda^a \mathrm{M} = \mathrm{E}^{-1}([0, a])$. We recall that there is a cap product pairing:

$$\mathrm{H}_*(\Lambda\mathrm{M}, \Lambda^0\mathrm{M}) \otimes \mathrm{H}^*(\Lambda\mathrm{M} - \Lambda^0\mathrm{M}) \xrightarrow{\cap} \mathrm{H}_*(\Lambda\mathrm{M}, \Lambda^0\mathrm{M})$$

(see [13], [14]). *We will say that two non zero homology classes* $z_1 \in \mathrm{H}_k(\Lambda\mathrm{M}, \Lambda^0\mathrm{M})$ *and* $z_2 \in \mathrm{H}_{k+j}(\Lambda\mathrm{M}, \Lambda^0\mathrm{M})$, $j > 0$, *are subordinated if there exists a non zero cohomology class* $w \in \mathrm{H}^j(\Lambda\mathrm{M} - \Lambda^0\mathrm{M})$ *such that* $z_1 = z_2 \cap w$. For a class $z \in \mathrm{H}_*(\Lambda\mathrm{M}, \Lambda^0\mathrm{M})$ we set:

$$\operatorname{minimax}(z) = \inf_{v \in z} \operatorname{minimax}(|v|)$$

where $v \in z$ means that v is a cycle in z and $|v|$ is its support.

THEOREM F. *Let z_1 and z_2 be subordinated homology classes in* $H_*(\Lambda M, \Lambda^0 M)$ *and* $k_i = minimax(z_i)$. *Then*:

1) k_i *is a critical value for* $\tilde{E}$,

2) $k_2 \geq k_1 > 0$,

3) *If* $k_1 = k_2$ *the set of critical points at* $\tilde{E}$*-level* k_1 *has covering dimension* $\geq \dim z_2 - \dim z_1$.

Theorem F is not yet completely satisfactory since, in case 3., if $\dim z_2 - \dim z_1 = 1$ the set of critical points at that level may just be the set of closed geodesics obtained rotating a given one.

We give now a description of the "S-equivariant theory". The circle S acts on ΛM by rotations and let us denote by ΠM the orbit's space. Since $\tilde{E}$, as well as the riemannian structure on ΛM, is invariant under the action of S, $\tilde{E}$ induces a continuous map $\tilde{E}_\Pi : \Pi M \to \mathbf{R}$ and φ_t induces a semigroup of $\tilde{E}_\Pi$-decreasing transformations $\varphi_t^\Pi : \Pi M \to \Pi$ Mcharacterized by the commutativity of the diagram:

$$\begin{array}{ccc} \Lambda M & \xrightarrow{\varphi_t} & \Lambda M \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \Pi M & \xrightarrow{\varphi_t^\Pi} & \Pi M \end{array}$$

where $\pi : \Lambda M \to \Pi M$ is the quotient map. As well as for ΛM we have a cap product pairing:

$$H_*(\Pi M, \Pi^0 M) \otimes H^*(\Pi M - \Pi^0 M) \xrightarrow{\cap} H_*(\Pi M, \Pi^0 M)$$

where $\Pi^a M = \tilde{E}_\Pi^{-1}([0, a])$, $a \geq 0$; therefore we also have the correspondent concept of subordinated homology classes.

If $A_\Pi \subseteq \Pi M$ is compact and not empty we define

$$\operatorname{minimax}(A_\Pi) = \lim_{t \to \infty} \max(\tilde{E}_\Pi |_{\varphi_t^\Pi(A_\Pi)})$$

and, for $z \in H_*(\Pi M, \Pi^0 M)$,

$$\operatorname{minimax}(z) = \inf_{v \in z} \operatorname{minimax}(|v|).$$

We have the analogous of theorem F for the new situation:

THEOREM G. *Let z_1 and z_2 be subordinated homology classes in* $H_*(\Pi M, \Pi^0 M)$ *and* $h_i = minimax(z_i)$. *Then*:

1) h_i *is a critical value for* $\tilde{E}$,

2) $h_2 \geq h_1 > 0$,

3) *If* $h_1 = h_2$ *the set in* ΠM *which is the image, under the quotient map, of the set of critical points of* $\tilde{E}$*-level* h_1 *in* ΛM, *has covering dimension* $\geq \dim z_2 - \dim z_1$.

Geodesics constructed with the method of theorems F *and* G *are called arithmetically distinct.* We want to point out explicitly that arithmetically distinct closed geodesics may not be geometrically distinct in the sense that they could be just coverings of the same geodesic. An important reason for considering the above concept is that if c_1 and c_2 are coverings of the same closed geodesic then the $\tilde{\mathrm{E}}$-value of c_i has to be relatively large. So the concept is particularly useful any time we can construct cycles in $(\Lambda\mathrm{M}, \Lambda^0\mathrm{M})$ or $(\Pi\mathrm{M}, \Pi^0\mathrm{M})$ with small minimaxes. Moreover there are non homogeneous variational problems that are treatable with the same methods. In this case, since the energy integral is not invariant for reparametrizations of the type $t \to at$, $a = \text{const.} > 0$, the concept of arithmetically distinct is most adequate.

5. The rational cohomology of $\Lambda\mathrm{M}$ and $\Pi\mathrm{M}$

We will describe now some of the properties of the cohomology of $\Lambda\mathrm{M}$ and $\Pi\mathrm{M}$ with rational coefficients (so $\mathrm{H}^*(\mathrm{X})$ will mean $\mathrm{H}^*(\mathrm{X}; \mathbf{Q})$). We will assume M compact and simply connected.

Sullivan has constructed cohomology classes $w_r \in \mathrm{H}^{ar+b}(\Lambda\mathrm{M})$, for some $a > 0$ and $b \geq 0$, that restrict non trivially to $\mathrm{H}^{ar+b}(\Omega\mathrm{M})$ (see [14], [25]). Since $\mathrm{H}^*(\Omega\mathrm{M})$ is a free algebra (in the sense of graded commutative and associative algebras) it follows that whenever an even dimensional class in $\mathrm{H}^*(\Lambda\mathrm{M})$ restricts non trivially to $\mathrm{H}^*(\Omega\mathrm{M})$, it generates a polynomial subalgebra in $\mathrm{H}^*(\Lambda\mathrm{M})$. This happens, for example, if the first non vanishing rational homotopy group is odd dimensional or if M has the homotopy type of the product of two compact manifolds.

Since the inclusion $\Lambda\mathrm{M} - \Lambda^0\mathrm{M} \hookrightarrow \Lambda\mathrm{M}$ is a homotopy equivalence (see [6]) it is easily seen that if $\mathrm{H}^*(\Lambda\mathrm{M})$ contains a polynomial algebra there are infinitely many subordinated homology classes in $\mathrm{H}_*(\Lambda\mathrm{M}, \Lambda^0\mathrm{M})$.

Unfortunally, in some interesting cases, as for example S^{2k}, the ring structure in $\mathrm{H}^*(\Lambda\mathrm{M})$ is not very rich. In these cases it is useful to look at $\mathrm{H}^*(\Pi\mathrm{M})$. The following theorem relates the two:

THEOREM H. *There is an exact sequence*:

$$\cdots \to \mathrm{H}^{r-1}(\Lambda\mathrm{M}, \Lambda^0\mathrm{M}) \to \mathrm{H}^{r-2}(\Pi\mathrm{M}, \Pi^0\mathrm{M}) \xrightarrow{\cup w_2}$$
$$\mathrm{H}^{r}(\Pi\mathrm{M}, \Pi^0\mathrm{M}) \xrightarrow{\Pi^*} \mathrm{H}^{r}(\Lambda\mathrm{M}, \Lambda^0\mathrm{M}) \to \cdots$$

where $w_2 \in \mathrm{H}^2(\Pi\mathrm{M} - \Pi^0\mathrm{M})$.

In particular we notice that the first non vanishing cohomology groups of $(\Lambda\mathrm{M}, \Lambda^0\mathrm{M})$ and $(\Pi\mathrm{M}, \Pi^0\mathrm{M})$ occur in the same dimension.

Example: Looking at the minimal model for ΛS^{2k} it is easily seen that:

$$H^s(\Lambda S^{2k}, \Lambda^0 S^{2k}) = \begin{cases} \mathbf{Q} & \text{if } s = (2j+1)(2k-1) \\ \mathbf{Q} & \text{if } s = (2j+1)(2k-1)+1 \\ 0 & \text{otherwise} \end{cases}$$

and the ring structure is the trivial one. In correspondence with a class z, generating $H^{2k-1}(\Lambda S^{2k}, \Lambda^0 S^{2k})$, there is a non vanishing class in $H^{2k-1}(\Pi S^{2k}, \Pi^0 S^{2k})$ such that the cup product with $(w_2)^p$ is non zero for, at least, $0 \leq p \leq 2k-1$ and so $H_*(\Pi S^{2k}, \Pi^0 S^{2k})$ contains at least $2k$ subordinated homology classes.

Another consequence of the previous theorem is the following:

THEOREM I. *At least one of the following facts holds*:

1) $H^*(\Lambda M)$ *contains a polynomial algebra*,

2) $H_*(\Pi M, \Pi^0 M)$ *contains two subordinated homology classes.*

Theorem I implies, in particular *the existence of at least two arithmetically distinct closed geodesics on any Finsler manifold* (*compact and simply connected*).

Remark. If the Finsler metric is symmetric, i.e. $F(X) = F(-X)$, instead of considering $\Pi M = \Lambda M/S$ we could consider $\overline{\Pi} M = \Lambda M/O(2)$ (theorem G holds also in this situation). In this case Klingenberg has proved the existence of, at least, three subordinated homology classes in $H_*(\overline{\Pi} M, \overline{\Pi}^0 M; Z_2)$. His proof does not carry over to $H_*(\Pi M, \Pi^0 M; Z_2)$. This justifies the assertion, made in the introduction, that the reason for Anosov's examples seems to be the non symmetry of the metric.

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