Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Haroon O. Tejumola

On a Lienard type matrix differential equation

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **60** (1976), n.2, p. 100–107.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1976_8_60_2_100_0>

Equazioni differenziali ordinarie. — *On a Lienard type matrix differential equation.* Nota di HAROON O. TEJUMOLA, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — L'Autore trova alcuni risultati sugli integrali di un'equazione n-vettoriale del tipo di Liénard.

1. INTRODUCTION

Let $\mathscr{M}$ denote the space of all real $n \times n$ matrices, $\mathscr{R}_n$ the real n-dimensional Euclidean space and $\mathscr{R}$ the real line. We shall consider here the differential equation

$$\ddot{X} + A\dot{X} + H(X) = P(t, X, \dot{X}) \tag{1.1}$$

where $X : \mathscr{R} \to \mathscr{M}$ is the unknown function; $A \in \mathscr{M}$ is a constant, $H : \mathscr{M} \to \mathscr{M}$ and $P : \mathscr{R} \times \mathscr{M} \times \mathscr{M} \to \mathscr{M}$. Systems of the type (1.1) occur in the theory of coupled circuits. Three specific properties of solutions of (1.1) will be examined, namely the stability of the trivial solution $X \equiv 0$ when $H(0) = 0$ and $P \equiv 0, 0 \in \mathscr{M}$, the ultimate boundedness of all solutions and the existence of periodic solutions, which properties are quite known for the special case in which (1.1) is an n-vector equation (so that $X : \mathscr{R} \to \mathscr{R}_n$, $H : R_n \to \mathscr{R}_n$ and $P : \mathscr{R} \times \mathscr{R}_n \times \mathscr{R}_n \to \mathscr{R}_n$); see [1], [2], [4] and [6]. Our present investigation are akin to those in [1] and [2], and our object is to provide extensions of some of the results therein to (1.1).

2. NOTATIONS AND DEFINITIONS

Some standard matrix notations will be used. For any $X \in \mathscr{M}$, X^t and $x_{ij}\, i, j = 1, 2, \cdots, n$ denote the transpose and the elements of X respectively while $(x_{ij})(y_{ij})$ will sometimes denote the product matrix XY of the matrices $X, Y \in \mathscr{M}$. $X_i = (x_{i_1}, x_{i_2}, \cdots, x_{i_n})$ and $X^j = \mathrm{col}(x_{1_j}, x_{2_j}, \cdots, x_{n_j})$ stand for the i^{th} row and j^{th} column of X respectively and $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ is the n^2 column vector consisting of the n rows of X.

We shall denote by JH(X) the $n^2 \times n^2$ generalized Jacobian matrix associated with the function $H : \mathscr{M} \to \mathscr{M}$ and evaluated at X: that is, JH(X) is the matrix associated with the Jacobian determinant $\dfrac{\partial(H_1, H_2, \cdots, H_n)}{\partial(X_1, X_2, \cdots, X_n)}$. Corresponding to the constant matrix $A \in \mathscr{M}$ we define an $n^2 \times n^2$ matrix $\tilde{A}$

(*) Nella seduta del 14 febbraio 1976.

consisting of n^2 diagonal $n \times n$ matrices $(a_{ij} \mathrm{I}_n)$ (I_n being the unit $n \times n$ matrix) and such that $(a_{ij} \mathrm{I}_n)$ belongs to the i^{th}-n row and j^{th}-n column (that is, counting n at a time) of $\tilde{\mathrm{A}}$.

Next we introduce an inner product $\langle \cdot , \cdot \rangle$ and a norm $\| \cdot \|$ on $\mathscr{M}$ as follows. For arbitrary $\mathrm{X}, \mathrm{Y} \in \mathscr{M}$, $\langle \mathrm{X}, \mathrm{Y} \rangle = \operatorname{trace} \mathrm{X}\mathrm{Y}^t$. It is easy to check that $\langle \mathrm{X}, \mathrm{Y} \rangle = \langle \mathrm{Y}, \mathrm{X} \rangle$ and that $\| \mathrm{X} - \mathrm{Y} \|^2 = \langle \mathrm{X} - \mathrm{Y}, \mathrm{X} - \mathrm{Y} \rangle$ defines a norm on $\mathscr{M}$. Indeed $\| \mathrm{X} \| = | \mathbf{X} |_{n^2}$ where $| \cdot |_{n^2}$ denotes the usual Euclidean norm in $\mathscr{R}_{n^2}$ and $\mathbf{X} \in \mathscr{R}_{n^2}$ is as defined above.

For any $\mathrm{X} \in \mathscr{M}$ an $n^2 \times n^2$ matrix $\mathrm{E}(\mathrm{X})$ will be said to be positive definite if

$$\mathbf{X}^t \, \mathrm{E}(\mathrm{X}) \, \mathbf{X} > 0 \qquad \text{for all} \quad \mathrm{X} \neq 0 \in \mathscr{M},$$

while $\mathrm{E}(\mathrm{X})$ is said to be strictly positive definite if there exists a constant $\delta > 0$ such that

$$\mathbf{X}^t \, \mathrm{E}(\mathrm{X}) \, \mathbf{X} \geq \delta \, | \mathbf{X} |_{n^2} \qquad \text{for all} \quad \mathrm{X} \in \mathscr{M}.$$

Lastly the symbol δ, with or without subscripts, denotes finite positive constants whose magnitudes depend only on A, H and P. Any δ, with a subscript, retains a fixed identity throughout while the unnumbered ones are not necessarily the same each time they occur.

3. STATEMENT OF RESULTS

Assume throughout the sequel that $\mathrm{H}(0) = 0$, $\mathrm{H} \in \mathrm{C}^1(\mathscr{M})$ and $\mathrm{P} \in \mathrm{C}(\mathscr{R} \times \mathscr{M} \times \mathscr{M})$. Our first result concerns the equation

$$\ddot{\mathrm{X}} + \mathrm{A}\dot{\mathrm{X}} + \mathrm{H}(\mathrm{X}) = 0. \tag{3.1}$$

THEOREM 1. *Let* H *satisfy a condition for the existence and uniqueness of solutions of* (3.1) *for any set of preassigned initial conditions. Suppose further that for arbitrary* $\mathrm{X} \in \mathscr{M}$

(i) *the matrices* $\tilde{\mathrm{A}}$ *and* $\mathrm{JH}(\mathrm{X})$ *are symmetric and,* $\tilde{\mathrm{A}}$ *commutates with* $\mathrm{JH}(\mathrm{X})$;

(ii) $\mathrm{JH}(\mathrm{X})$ *is strictly positive definite and the product matrix* $\tilde{\mathrm{A}}\mathrm{JH}(\mathrm{X})$ *is positive definite. Then every solution* $\mathrm{X}(t)$ *of* (3.1) *satisfies*

$$\| \mathrm{X}(t) \| \to 0 \qquad \textit{and} \quad \| \dot{\mathrm{X}}(t) \| \to 0 \qquad \textit{as} \quad t \to \infty. \tag{3.2}$$

This result is a matrix analogue of [2; Corollary].

For the next two results, it will be further assumed that H and P satisfy a condition for the existence of solutions of (1.1) for any set of preassigned initial conditions.

THEOREM 2. *Suppose that hypothesis* (i) *of Theorem 1 and the following conditions hold*:

(i) $\tilde{A}$ *is positive definite, and the product matrix* $\tilde{A}JH(X)$ *is strictly positive definite for* $\|X\| \geq \rho > 0$, ρ *a constant*;

(ii) *for all* t *and arbitrary* $X, Y \in \mathcal{M}$, P *satisfies*

$$\|P(t, X, Y)\| \leq \Delta_0 + \varepsilon(\|X\| + \|Y\|), \tag{3.3}$$

where $\Delta_0 \geq 0$, $\varepsilon \geq 0$ *are constants and* ε *is sufficiently small. Then every solution* $X(t)$ *of* (1.1) *satisfies*

$$\|X(t)\| \leq \Delta_1 \quad , \quad \|\dot{X}(t)\| \leq \Delta_1 \tag{3.4}$$

for all t *sufficiently large, where* $\Delta_1 > 0$ *is a constant whose magnitude depends only on* $\Delta_0, \varepsilon, A, H$ *and* P.

This result provides an extension of [1; Theorem 2].

THEOREM 3. *Suppose, further to the condition of Theorem 2, that* P *satisfies*

$$P(t, X, Y) = P(t + \omega, X, Y)$$

for all $X, Y \in \mathcal{M}$. *Then* (1.1) *admits of at least one* ω-*periodic solution.*

4.

We require the following subsidiary result:

LEMMA. *Assume that hypothesis* (i) *of Theorem 1 holds. Then*

$$\langle H(X), AX\rangle = \int_0^1 \mathbf{X}^t \tilde{A}\, JH(\sigma X)\, \mathbf{X}\, d\sigma; \tag{4.1}$$

$$\frac{d}{dt}\int_0^1 \langle H(\sigma H), X\rangle\, d\sigma = \langle H(X), \dot{X}\rangle \qquad \textit{for all} \quad X \in \mathcal{M}. \tag{4.2}$$

Proof. Since each $h_{ij} \in C^1(\mathcal{M})$, $i, j = 1, 2, \cdots, n$, it is clear that

$$h_{ij}(X) = \int_0^1 \frac{d}{d\sigma} h_{ij}(\xi)\, d\sigma = \int_0^1 \sum_{k,l=1}^{n} \frac{\partial h_{ij}}{\partial x_{kl}}(\xi)\, x_{kl}\, d\sigma \quad , \quad \xi = \sigma X$$

and, by the definition of inner product,

$$\langle H(X), AX\rangle = \int_0^1 \sum_{i,j=1}^{n} \sum_{k,l=1}^{n} \frac{\partial h_{ij}}{\partial x_{kl}}(\xi)\, x_{kl} \sum_{k=1}^{n} a_{ik}\, x_{kj}\, d\sigma.$$

Since $\tilde{\mathrm{A}}$ is symmetric, the representation (4.1) now follows from the definitions of $\tilde{\mathrm{A}}$ and JH (X).

To verify (4.2) observe that

$$(4.4) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \langle \mathrm{H}(\sigma \mathrm{X}), \mathrm{X}\rangle = \int_0^1 \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}(\sigma \mathrm{X}), \mathrm{X}\right\rangle \mathrm{d}\sigma + \int_0^1 \langle \mathrm{H}(\sigma \mathrm{X}), \dot{\mathrm{X}}\rangle \,\mathrm{d}\sigma ;$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}(\sigma \mathrm{X}) = \left(\sigma \sum_{k,l=1}^{n} \frac{\partial h_{ij}}{\partial x_{kl}}(\xi)\,\dot{x}_{kl}\right) \quad , \qquad \xi = \sigma \mathrm{X}\,.$$

Thus, by the definition of inner product and since JH (X) is symmetric,

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}(\sigma \mathrm{X}), \mathrm{X}\right\rangle = \sigma \sum_{i,j=1}^{n}\left(\sum_{k,l=1}^{n} \frac{\partial h_{ij}}{\partial x_{kl}}(\xi)\,\dot{x}_{kl}\right) x_{ij} =$$

$$= \sigma \sum_{i,j=1}^{n}\left(\sum_{k,l=1}^{n} \frac{\partial h_{kl}}{\partial x_{ij}}(\xi)\,\dot{x}_{kl}\right) x_{ij}$$

and, by interchanging the order of summation and replacing k, l with i and j respectively, we have that

$$(4.5) \qquad \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}(\sigma \mathrm{X}), \mathrm{X}\right\rangle = \sigma \sum_{i,j=1}^{n}\left(\sum_{k,l=1}^{n} \frac{\partial h_{ij}}{\partial x_{kl}}(\xi)\,x_{kl}\right) \dot{x}_{ij} =$$

$$= \left\langle \sigma \frac{\mathrm{d}}{\mathrm{d}\sigma}\mathrm{H}(\sigma \mathrm{X}), \dot{\mathrm{X}}\right\rangle \cdot$$

The result (4.2) now follows on integrating (4.5) by parts and using (4.4).

5. PROOF OF THEOREM 1

It will be convenient to replace (3.1) by the system

$$(5.1) \qquad \dot{\mathrm{X}} = \mathrm{Y} \quad , \quad \dot{\mathrm{Y}} = -\mathrm{AY} - \mathrm{H}(\mathrm{X})\,.$$

We shall show that every solution (X, Y) of (5.1) satisfies

$$(5.2) \qquad \|\mathrm{X}(t)\| \to 0 \quad \text{and} \quad \|\mathrm{Y}(t)\| \to 0 \quad \text{as} \quad t \to \infty\,.$$

Consider the function $\mathrm{V} : \mathscr{M} \times \mathscr{M} \to \mathscr{R}$ defined by

$$(5.3) \qquad 2\,\mathrm{V} = \langle \mathrm{AX} + \mathrm{Y}, \mathrm{AX} + \mathrm{Y}\rangle + 2\int_0^1 \langle \mathrm{H}(\sigma \mathrm{X}), \mathrm{X}\rangle \,\mathrm{d}\sigma$$

$$\equiv \qquad \mathrm{V}_1 \qquad\qquad + 2\,\mathrm{V}_2\,,$$

where, by the definition of inner product and by (4.1) with $A \equiv I_n$,

$$V_1 = \sum_{i=1}^{n} |AX^i + Y^i|_{n^2} \geq 0 ;$$

$$V_2 = \int_0^1 \sigma \int_0^1 \mathbf{X}^t \, JH (\tau\sigma \, X) \, \mathbf{X} \, d\tau \, d\sigma$$

$$\geq \delta_0 \, |\mathbf{X}|_{n^2}^2 = \delta_0 \sum_{i=1}^{n} |X^i|_n^2 , \qquad \text{for all} \quad X , Y \in \mathcal{M},$$

since JH (X) is strictly positive definite. Therefore V satisfies

$$(5.4) \qquad 2\,V \geq \sum_{i=1}^{n} (|AX^i + Y^i|_n^2 + \delta_0 \, |X^i|_n^2) .$$

Now let (X , Y) be any solution of (5.1). Then, from (5.1) and (4.2),

$$\dot{V} = - \langle H(X) , AX \rangle ,$$

so that, by (4.1) and the strict positiveness of $\tilde{A}JH(X)$,

$$(5.5) \qquad \dot{V} < 0 \quad \text{if} \quad X \neq 0 \quad \text{and} \quad \dot{V} \leq 0 \quad \text{for all} \quad (X , Y).$$

Let $\mathbf{M} = \mathcal{M} \times \mathcal{M}$ be the product space equipped with the metric d defined, for any pair of points $Q = (X , Y) , R = (U , V) \in \mathbf{M}$, by

$$d(Q , R) = \|X - U\| + \|Y - V\| .$$

It is easy to see that the solutions (X , Y) of (5.1) give rise to a dynamical system in **M**. The notions of a trajectory and positive half trajectory issuing from a point of **M**, and of ω-limit points can be defined in the usual way. By using a modified form of the arguments in [5], it can also be shown that the results (5.4) and (5.5) imply (5.2). Further details of the arguments will be omitted here.

6. Proof of Theorem 2

Consider the function $V : \mathcal{M} \times \mathcal{M} \to \mathcal{R}$ adapted from [1], and defined for any $X , Y \in \mathcal{M}$ by

$$(6.1) \qquad 2\,V = \langle AX + 2\,Y , AX + 2\,Y \rangle + \langle AX , AX \rangle + \delta_2 \int_0^1 \langle H(\sigma X) , X \rangle \, d\sigma$$

$$\equiv \qquad \theta_1 \; + \qquad\qquad + \delta_2 \, \theta_3 .$$

It is clear that

$$\theta_1 = \sum_{i=1}^{n} |AX^i + Y^i|_n^2 \quad , \quad \theta_2 = \sum_{i=1}^{n} |AX^i|_n^2$$

and, since A is assumed positive definite, there exists δ_3 such that

$$(6.2) \qquad \theta_1 + \theta_2 \geq \delta_3 \sum_{i=1}^{n} (|X^i|_n^2 + |Y^i|_n^2) = \delta_3 (\|X\|^2 + \|Y\|^2).$$

As for therm θ_3, observe from the Lemma with $A \equiv I_n$ that

$$(6.3) \qquad \theta_3 = \int_0^1 \sigma \int_0^1 \mathbf{X}^t \, JH(\sigma\tau X) \, \mathbf{X} \, d\tau \, d\sigma .$$

The double integral (6.3) can be evaluated in much the same way as 2.3 (13) of [3] to yield the result

$$\int_0^1 \langle H(\sigma X), X \rangle \, d\sigma \geq -\delta \qquad \text{for all} \quad X \in \mathcal{M}.$$

This, combined with (6.1) and (6.2) shows that

$$(6.4) \qquad 2V \geq \delta_3 (\|X\|^2 + \|Y\|^2) - \delta \qquad \text{for all} \quad X, Y \in \mathcal{M} .$$

It is convenient now to consider (1.1) in the system form

$$(6.5) \qquad \dot{X} = Y \quad , \quad \dot{Y} = -AY - H(X) + P(t, X, Y).$$

On differentiating (6.1) and using (4.2), we have that

$$(6.6) \qquad \dot{V} = -2 \langle H(X), AX \rangle - 2 \langle AY, Y \rangle + 2 \langle P(t, X, Y), AX + 2Y \rangle .$$

Since $H \in C'(\mathcal{M})$, we have from (4.1) and hypothesis (i) of Theorem 2 that

$$\langle H(X), AX \rangle \geq -\delta \qquad \text{if} \quad \|X\| \leq \rho .$$

On the other hand, if $\|X\| \geq \rho$, then

$$\langle H(X), AX \rangle = \int_0^{\rho/\|X\|} \mathbf{X}^t \tilde{A} JH(\sigma X) \, \mathbf{X} \, d\sigma + \int_{\rho/\|X\|}^1 \mathbf{X}^t \tilde{A} JH(\sigma X) \, \mathbf{X} \, d\sigma$$

$$\equiv \quad I_1 + I_2 .$$

In I_1, $\|\sigma X\| \leq \rho$ and so for some δ, $\|\mathbf{X}^t \tilde{A} JH(\sigma X) \, \mathbf{X} \, d\sigma\| \leq \delta \|X\|^2$; hence

$$|I_1| \leq (\rho/\|X\|) \, \delta \|X\|^2 = \rho\delta \|X\| .$$

In I_2, $\|\sigma X\| \geq \rho$ so that for some δ_4,

$$I_2 \geq \delta_4 \|X\|^2 (1 - \rho/\|X\|) = \delta_4 (\|X\|^2 - \rho \|X\|),$$

since $JH(X)$ is strictly positive definite if $\|X\| \geq \rho$. Thus, for all $X \in \mathcal{M}$,

$$2 \langle H(X), AX \rangle \geq 2 \delta_4 \|X\|^2 - \delta (\|X\| + 1).$$

Lastly, since A is positive definite and P satisfies (3.3), we have that

$$\langle AY , Y\rangle = \mathbf{Y}^t \tilde{A}\mathbf{Y} \geq \delta_5 \| Y \|^2 ;$$

$$2 |\langle P , AX + 2 Y\rangle| \leq \varepsilon \delta_6 (\| X \|^2 + \| Y \|^2) + \delta_7 (\| X \| + \| Y \|)$$

for some $\delta_5 , \delta_6 , \delta_7$. The various estimates combined with (6.6) imply

$$\dot{V} \leq -(2 \delta_8 - \varepsilon \delta_6) (\| X \|^2) + \| Y \|^2 + \delta (\| X \| + \| Y \| + 1) ,$$

where $\delta_8 = \min (\delta_4 , \delta_5)$. Thus, if ε is fixed such that

$$\varepsilon \leq \delta_8 \delta_6^{-1} \quad , \quad \delta_8 \equiv \min (\delta_4 , \delta_5)$$

(and this is assumed henceforth) then there exists δ_9 such that

$$\dot{V} \leq -1 \quad \text{if} \quad \| X \|^2 + \| Y \|^2 \geq \delta_9^2 . \tag{6.7}$$

The result (3.4) now follows in the usual way from (6.4) and (6.7).

7. Proof of Theorem 3

The proof is by the Leray-Schauder fixed point technique. Consider the parameter μ-dependent equation

$$\ddot{X} + A\dot{X} + (1 - \mu) \delta_{10} X + \mu H (X) = \mu P (t , X , \dot{X}) , \qquad 0 \leq \mu \leq 1, \tag{7.1}$$

where δ_{10} is fixed (as is possible, since JH (X) is assumed strictly positive definite) such that

$$\mathbf{X}^t JH (X) \mathbf{X} \geq \delta_{10} \| X \|^2 \quad \text{if} \quad \| X \| \geq \rho . \tag{7.2}$$

The equation (7.1) reduces to the linear equation

$$\ddot{X} + A\dot{X} + \delta_{10} X = 0 \tag{7.3}$$

when $\mu = 0$ and to the original equation (1.1) when $\mu = 1$. Since the trivial solution of (7.3) is, in view Theorem 1, asymptotically stable in the large, a modified form of the arguments in [3, § 4] would yield Theorem 3 once it is shown that solutions of (7.1) are ultimately bounded with bounding constant independent of μ. The actual verification of this boundedness property is as in § 6, since the Jacobian matrix

$$JH_\mu (X) = (1 - \mu) \delta_{10} I_{n^2} + \mu JH (X)$$

of $H_\mu (X) = (1 - \mu) \delta_{10} X + \mu H (X)$ satisfies all of the hypotheses of Theorem 2 if JH (X) does. Further details of the proof will be omitted here.

Remarks. Our present investigation is of an exploratory nature, efforts are being made to expand its scope to cover the situation in which the unknown function X in (1.1) is not necessarily a square matrix. Our results in this direction will be announced elsewhere.

REFERENCES

[1] J. O. C. EZEILO (1966) – « Ann. Mat. Pura. Appl. (IV) », *72*, 239–252.
[2] J. O. C. EZEILO (1966) – « Ann. Mat. Pura. Appl. (IV) », *73*, 17–26.
[3] J. O. C. EZEILO and H. O. TEJUMOLA (1966) – « Ann. Mat. Pura Appl. (IV) », *74*, 283–316.
[4] S. MIZOHATA (1952) – « Mem. Coll. Sci. Univ. Kyoto, Ser. A. Mat. », *27*, 115–121.
[5] V. V. NEMYTSKII and V. V. STEPANOV (1960) – *Qualitative Theory of Differential Equations*, Princeton University Press.
[6] R. A. SMITH (1963) – « Rendiconti del Circolo Mat. di Palermo », ser. II, *12*, 1–18.
[7] R. A. SMITH (1969) – « Ann. Mat. Pura Appl. (IV) », *82*, 257–266.