Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Adolf Haimovici

On the behaviour of the solutions of a system of differential equations with set functions as unknowns. Nota II

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **60** (1976), n.1, p. 32–41.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1976_8_60_1_32_0>

Equazioni funzionali. — *On the behaviour of the solutions of a system of differential equations with set functions as unknowns.* Nota II di ADOLF HAIMOVICI, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Ved. quello dato al principio della Nota I, apparsa nel precedente fascicolo di questi « Rendiconti », alla fine della quale trovasi la Bibliografia. Le numerazioni dei paragrafi e delle formule continuano quelle di detta Nota I.

4. THE LINEAR SYSTEM

Consider the linear system

$$\frac{d\varphi}{d\mu}(x) = A\varphi(P_x), \qquad \varphi = (\varphi_1, \varphi_2, \cdots, \varphi_m), \tag{18}$$

where A is, as in (9), a constant matrix. We look for a solution of (18), with singular part $\nu = (\nu_1, \nu_2, \cdots, \nu_m)$ with support on the null measure set $H \in R^k$.

Let S be a linear transformation, with non-singular matrix (s_{ij}), and

$$\psi = S\varphi \qquad \det S \neq 0, \qquad \psi = (\psi_1, \psi_2, \cdots, \psi_m), \tag{19}$$

chosen so that $SAS^{-1} = J$ is a Jordan canonical matrix. Equation (18) becomes:

$$\frac{d\psi}{d\mu}(x) = J\psi, \tag{20}$$

where

$$J = \begin{pmatrix} \Lambda_1 & 0 & 0 & \cdots \\ 0 & \Lambda_2 & 0 & \cdots \\ 0 & 0 & \Lambda_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \quad \Lambda_i = \begin{pmatrix} \Lambda_{i1} & 0 & 0 & \cdots \\ 0 & \Lambda_{i2} & 0 & \cdots \\ 0 & 0 & \Lambda_{i3} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \quad \Lambda_{ij} = \begin{pmatrix} \lambda_i & 0 & 0 & \cdots \\ 1 & \lambda_i & 0 & \cdots \\ 0 & 1 & \lambda_i & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Denote also by

$$h = S\nu, \qquad h = (h_1, h_2, \cdots, h_m) \tag{21}$$

the singular measures of ψ_i, with the same support H as φ_i. If λ_i is a real eigenvalue of A, the subsystem of (20) corresponding to it will be

$$\begin{aligned} \frac{d\psi_{i1}}{d\mu}(x) &= \lambda_i \psi_{i1}(P_x) \\ \frac{d\psi_{i2}}{d\mu}(x) &= \psi_{i1}(P_x) + \lambda_i \psi_{i2}(P_x) \\ &\cdots\cdots\cdots\cdots \\ \frac{d\psi_{1,s_i}}{d\mu}(x) &= \psi_{i,s_i-1}(P_x) + \lambda_i \psi_{i,s_i}(P_x) \end{aligned} \tag{22}$$

(*) Nella seduta del 10 gennaio 1976.

(we have used a notation with two indices for the components of ψ: the first index labels the eigenvalue, the second one denotes the rank of the component in the set of components with the same λ_i; an analogous notation will be used in the following).

If there exist also complex eigenvalues of A, then we shall perform a second linear transformation

$$\Phi = T\psi = TS\varphi ,$$

letting unchanged the ψ's corresponding to the real λ's, and changing the others, corresponding to a pair of complex ones $\lambda, \lambda_{j+1} = \bar{\lambda}_j$ in

$$(32) \qquad \begin{aligned} \Phi_{j,l}(P_x) &= \frac{1}{2}\{\psi_{jl}(P_x) + \psi_{j+1,l}(P_x)\} , \\ \Phi_{j+1,l}(P_x) &= \frac{1}{2i}\{\psi_{jl}(P_x) - \psi_{j+1,l}(P_x)\} . \end{aligned}$$

Denote by I the set of indices i, for which λ_i is real, and by J the set of indices j for which λ_j is complex. The system in Φ will be:

$$(24_1) \qquad \left\{ \begin{aligned} &\frac{d\Phi_{i1}}{d\mu}(x) = \lambda_i \Phi_{i1}(P_x) \\ &\frac{d\Phi_{i2}}{d\mu}(x) = \Phi_{i1}(P_x) + \lambda_i \Phi_{i2}(P_x) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\frac{d\Phi_{i,s_i}}{d\mu}(x) = \Phi_{i,s_i-1}(P_x) + \lambda_i \Phi_{i,s_i}(P_x) , \end{aligned} \right. \qquad (i \in I)$$

$$(24_2) \qquad \left\{ \begin{aligned} &\frac{d\Phi_{j1}}{d\mu}(x) = a_j \Phi_{j1}(P_x) - b_j \Phi_{j+1,1}(P_x) \quad , \quad \lambda_j = a_j + ib_j \\ &\frac{d\Phi_{j2}}{d\mu}(x) = a_j \Phi_{j2}(P_x) - b_j \Phi_{j+1,2}(P_x) + \Phi_{j1}(P_x) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\frac{d\Phi_{j,s_j}}{d\mu}(x) = a_j \Phi_{j,s_j}(P_x) - b_j \Phi_{j+1,s_j}(P_x) + \Phi_{j,s_j-1} , \\ &\frac{d\Phi_{j+1,1}}{d\mu}(x) = a_j \Phi_{j+1,1}(P_x) + b_j \Phi_{j,1}(P_x) \\ &\frac{d\Phi_{j+1,2}}{d\mu}(x) = a_j \Phi_{j+1,2}(P_x) + b_j \Phi_{j,2}(P_x) + \Phi_{j+1,1}(P_x) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\frac{d\Phi_{j+1,s_1}}{d\mu}(x) = a_j \Phi_{j+1,s_1}(P_x) + b_j \Phi_{j,s_1}(P_x) + \Phi_{j+1,s_1-1}(P_x). \end{aligned} \right. \qquad (j \in J)$$

The singular measures of Φ_i will now be:

(25) $$\pi = \mathrm{T}h = \mathrm{TS}\,\nu\,, \qquad \pi = (\pi_1, \pi_2, \cdots, \pi_m)\,,$$

with the same support H. In the real case $\pi_i = h_i$, in the complex one we shall write $\pi_j = p_j + iq_j$.

Taking into account the properties of the functions $w(x, y, \lambda)$, $w_t(x, y, \lambda)$, $\mathrm{V}_t(x, y, \lambda)$ and $\mathrm{W}_t(x, y, \lambda)$, defined in the previous paragraph, the solution of this system will be:

(26) $$\Phi_{ir}(\mathrm{P}_x) = \int_{\mathrm{H}} \sum_{l=1}^{r} w_l(x, y, \lambda_i)\, \mathrm{d}m_{i,r-l+1,y} \qquad (i \in \mathrm{I}\,,\ r = 1, 2, \cdots, s_i)$$

(26$_1$) $$\Phi_{jr}(\mathrm{P}_x) = \int_{\mathrm{H}} \sum_{l=1}^{r} \{\mathrm{V}_{r-l+1}(x, y, \lambda_j)\, \mathrm{d}p_{j,l,y} - \mathrm{W}_{r-l+1}(x, y, \lambda_j)\, \mathrm{d}q_{jl,y}\}$$

$$\Phi_{j+1,r}(\mathrm{P}_x) = \int_{\mathrm{H}} \sum_{l=1}^{r} \{\mathrm{V}_{r-l+1}(x, y, \lambda_j)\, \mathrm{d}q_{jl,y} + \mathrm{W}_{r-l+1}(x, y, \lambda_j)\, \mathrm{d}p_{jl,y}\}.$$

After this preparation , we can prove

THEOREM A. *Suppose that*:

i) *In the complex plane there exists a domain* Σ, *such that, if* $\lambda \in \Sigma$ *then*

(27) $$|w_t(x, y, \lambda)| \leq \mathrm{M} = \text{const.} \qquad (t = 1, 2, \cdots);$$

ii) *all the eigenvalues of* A *are in* Σ;

then the trivial solution of (18) *is stable, in the sense of our definition.*

Proof. Starting with (18), we perform a linear transformation $\mathrm{U} = \mathrm{TS}$ (as in the previous considerations) and arrive to system (24$_1$), (24$_2$) the solution of which is (26), (26$_1$). Between the functions φ and Φ, the two sets of inequalities hold:

(28) $$a^2 \sum_{i=1}^{m} \Phi_i^2 \leq \sum_{i=1}^{m} \varphi_i^2 \leq \mathrm{A}^2 \sum_{i=1}^{m} \Phi_i^2 \quad , \quad a'^2 \sum_{i=1}^{m} \varphi_i^2 \leq \sum_{i=1}^{m} \Phi_i^2 \leq \mathrm{A}'^2 \sum_{i=1}^{m} \varphi_i^2\,,$$

with suitable constants a, a', A, A$'$ depending on the elements of the matrix U.

Remark now that from (27) it follows

(29) $$|\mathrm{V}_j(x, y, \lambda)| \quad , \quad |\mathrm{W}_t(x, y, \lambda)| \leq \mathrm{M}\,.$$

From the expression (26), (26$_1$) of the solution, it easily follows

$$|\Phi_{ir}(\mathrm{P}_x)| \leq \mathrm{M} \int_{\mathrm{H}} \sum_{l=1}^{r} |\mathrm{d}m_{i,r-l+1}| \qquad (i \in \mathrm{I})\,,$$

$$|\Phi_{jr}(\mathrm{P}_x)| \leq \mathrm{M} \int_{\mathrm{H}} \sum_{l=1}^{r} (|\mathrm{d}p_{jl}| + |\mathrm{d}q_{jl}|) \qquad (j \in \mathrm{J})\,.$$

Choose now h, p, q such that:

(30) $$\int_H |dh_{i,r}| \leq \eta \quad , \quad \int_H |dp_{jr}| \quad , \quad \int_H |dq_{jr}| \leq \eta/2 \; .$$

It follows:

(31) $$\sum_{i\in I} \sum_{r=1}^{s_i} (\Phi_{ir})^2 + \sum_{j\in J} \sum_{j=1}^{s_j} (\Phi_{jr})^2 \leq M^2 C^2 \eta^2 ,$$

with a constant C easily computable. Using (28), it follows $\sum_{i=1}^{m} \varphi_i^2 \leq A^2 M^2 C^2 \eta^2$.

If we take $\eta \leq (ACM)^{-1} \varepsilon$, it follows $\Sigma\varphi_i^2 \leq \varepsilon^2$.

It remains to be proved that we can choose the singular measures in such a way that (30) is fulfilled; now, from inequalities of the same type as (27), and taking into account that $\pi = TS\nu = U\nu$, we deduce

$$\sum_{i\in I} \sum_{r=1}^{s_i} (h_{ir})^2 + \sum_{j\in J} \sum_{l=1}^{s_j} ((p_{il})^2 + (q_{il})^2) \leq \frac{\eta}{a^2} \sum \nu_i^2.$$

Take ν_i such that $\nu_i \leq \eta/(mb)$, where b is the greatest absolute value of the coefficients of the matrix $U^{-1} = S^{-1} T^{-1}$; we get $\pi_i = \sum_{j=1}^{m} b_{ij} \nu_j$ and $|\pi_i| \leq \eta$. It follows then that, if we take

(32) $$|\nu_i| \leq \frac{1}{mb} (ACM)^{-1} \epsilon ,$$

the relation (31) is a consequence; and so the theorem is proved.

5. NON LINEAR SYSTEMS

We now come back to system (9), and suppose:

a) the matrix A has its eigenvalues λ_i in the domain Σ, where for $x \in \Omega, y \in H, w_t(x, y, \lambda)$ are bounded;

b) the functions $G_i(x, P_x)$ defined on $\Omega \times R^m$ satisfy uniqueness conditions for system (9), and

$$G_i(x, 0) = 0;$$

c) the same functions G_i satisfy also

(33) $$\|G(x, \varphi(P_x))\| \leq K(x) \left(\sum_{i=1}^{m} (\varphi_i(P_x))^2 \right)^{(1+\alpha)/2} ,$$

where $K(x)$ satisfies

(34) $$\sigma(\sqrt{m})^{1+\alpha} \int_{P_x} K(y) (\omega(y, 1 + L))^\alpha d\mu_y \leq M_1 < +\infty$$

(σ is the greatest order of multiplicity of the eigenvalues of A, and L is the greatest absolute value of the eigenvalues of A).

THEOREM B. *If* A , H , G *satisfy the above conditions* $a)$, $b)$, $c)$ *and the mapping* P *satisfies conditions* $i)$, $ii)$ of § 2 *and* $\gamma)$ *of* § 3, *then the trivial solution of* (9) *is stable, in the sense of the definition of* § 1.

Proof. Let be $\varepsilon > 0$ and ν_i the singular measures of φ_i, with support on the null-measure set H. Applying the transformation U = TS to the vector function φ, we obtain the system:

$$(35_1)\qquad \begin{cases} \dfrac{d\Phi_{i1}}{d\mu}(x) = \lambda_i\,\Phi_{i1}(P_x) + F_{i1}(x\,,\Phi(P_x)) \\[2mm] \dfrac{d\Phi_{i2}}{d\mu}(x) = \Phi_{i1}(P_x) + \lambda_i\,\Phi_{i2}(P_x) + F_{i2}(x\,,\Phi(P_x)) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{d\Phi_{is_i}}{d\mu}(x) = \Phi_{i,s_i-1}(P_x) + \lambda_i\,\Phi_{i,s_i}(P_x) + F_{i,s_i}(x\,,\Phi(P_x)) \end{cases} \qquad (i\in \mathrm{I}),$$

$$(35_2)\qquad \begin{cases} \dfrac{d\Phi_{j1}}{d\mu}(x) = a_j\,\Phi_{j1}(P_x) - b_j\,\Phi_{j+1,1}(P_x) + F_{j1}(x\,,\Phi(P_x)) \\[2mm] \dfrac{d\Phi_{j2}}{d\mu}(x) = a_j\,\Phi_{j2}(P_x) - b_j\,\Phi_{j+1,2}(P_x) + \Phi_{j,1}(P_x) + \\ \qquad + F_{j,2}(x\,,\Phi(P_x)) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{d\Phi_{j,s_j}}{d\mu}(x) = a_j\,\Phi_{j,s_j}(P_x) - b_j\,\Phi_{j+1,s_j}(P_x) + \Phi_{j,s_j-1}(P_x) + \\ \qquad + F_{j,s_j}(x\,,\Phi(P_x)) \\ \dfrac{d\Phi_{j+1,1}}{d\mu}(x) = b_j\,\Phi_{j1}(P_x) + a_j\,\Phi_{j+1,1}(P_x) + F_{j+1,1}(x\,,\Phi(P_x)) \\[2mm] \dfrac{d\Phi_{j+1,2}}{d\mu}(x) = b_j\,\Phi_{j2}(P_x) + a_j\,\Phi_{j+1,2}(P_x) + \Phi_{j+1,1}(P_x) + \\ \qquad + F_{j+1,1}(x\,,\Phi(P_x)) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{d\Phi_{j+1,s_j}}{d\mu}(x) = b_j\,\Phi_{js_j}(P_x) + a_j\,\Phi_{j+1,s_j}(P_x) + \Phi_{j+1,s_j-1}(P_x) + \\ \qquad + F_{j+1,s_j}(x\,,\Phi(P_x)) \end{cases} \qquad (j\in J),$$

where we have used the same notation as in § 2, and have denoted F = UG.

The singular measure of φ will be, as in § 2, $\pi = \mathrm{U}\nu$, and we denote $h_i = \pi_i$ for $i \in \mathrm{I}$, and $\pi_j = p_j + iq_j$ for $j \in \mathrm{I}$.

The hypothesis $b)$ for G, leads to $F(x\,,0) = 0$, and $c)$ leads to:

$$(36)\qquad \|F(x\,,\Phi)\| \le \tilde{K}(x)\left(\sum_{i=1}^{m} \Phi_i^2\right)^{(1+\alpha)/2},$$

where $\tilde{K}(x)$ differs from $K(x)$ by a multiplicative constant, and consequently satisfies a condition analogous to (33):

$$\sigma m^{(1+\alpha)/2} \int_{P_x} \tilde{K}(x)(y, 1 + L))^\alpha \, d\mu_y \leq M_2 < +\infty . \tag{37}$$

The solution of (35_1), (35_2) is given by the intermediate of the Volterra integral equations system:

$$\Phi_{ir}(P_x) = \Phi^0_{ir}(P_x) + \int_{P_x} \sum_{l=1}^{r} w_l(x, y, \lambda_i) F_{i,r-l+1}(y, \Phi(P_y)) \, d\mu_y \qquad (i \in I),$$

(38)

$$\Phi_{j,r}(P_x) = \Phi^0_{jr}(P_x) + \int_{P_x} \sum_{l=1}^{r} \{V_l(x, y, \lambda_j) F_{j,l-r+1}(y, \Phi(P_y)) - W_l(x, y, \lambda_j) F_{j+1,l-r+1}(y, \Phi(P_y))\} \, d\mu_y \qquad (j \in J)$$

$$\Phi_{j+1,r}(P_x) = \Phi^0_{j+1,r}(P_x) + \int_{P_x} \sum_{l=1}^{r} \{W_l(x, y, \lambda_j) F_{j+1,l-r+1}(y, \Phi(P_y)) + W_l(x, y, \lambda_j) F_{j,l-r+1}(y, \Phi(P_y))\} \, d\mu_y ,$$

where $\Phi^0(P_x)$ is the solution of the linear system (24), with singular part π.

The equivalence between the stated problem and system (38) can be proved by differentiation of the two members of the last equations, taking into account the differentiation formula given in § 3 and relations (14).

We shall now prove that the singular parts of Φ_i can be chosen so that

$$\sum_{i=1}^{m} (\Phi_i(P_x))^2 < \eta_1^2 , \tag{39}$$

η_1 being a given real number. To this end, consider the operator $\mathcal{I}$ defined by $\tilde{\Phi}(P_x) = (\mathcal{I}\Phi)(P_x)$, where $\mathcal{I}\Phi$ is given by

(40)

$$\tilde{\Phi}_{ir}(P_x) = \Phi^0_{ir} + \int_{P_x} \sum_{l=1}^{r} w_l(x, y, \lambda_i) F_{i,r-l+1}(y, \Phi(P_y)) \, d\mu_y \qquad (i \in I)$$

$$\tilde{\Phi}_{jr}(P_x) = \Phi^0_{jr} + \int_{P_x} \sum_{l=1}^{r} \{V_l(x, y, \lambda_j) F_{j,r-l+1}(y, \Phi(P_y)) - W_l(x, y, \lambda_j) F_{j,l-r+1}(y, \Phi(P_y))\} \, d\mu_y ,$$

$$\tilde{\Phi}_{j+1,r}(P_x) = \Phi^0_{j+1,r}(P_x) + \int_{P_x} \sum_{l=1}^{r} \{V_l(x, y, \lambda_j) F_{j+1,l-r+1}(y, \Phi(P_y)) + W_l(x, y, \lambda_j) F_{j,l-r+1}(y, \Phi(P_y))\} \, d\mu_y .$$

Let η be a positive, for the moment arbitrary number, and $\mathcal{M}$ the set of vector functions Φ satisfying:

$$\Sigma \left(\frac{\Phi_i (\mathrm{P}_x)}{\omega (x , 1 + \mathrm{L})} \right)^2 < \eta^2 , \tag{41}$$

and write (40) under the form:

$$\frac{\tilde{\Phi}_{ir} (\mathrm{P}_x)}{\omega (x , 1 + \mathrm{L})} = \frac{\Phi^0_{ir} (\mathrm{P}_x)}{\omega (x , 1 + \mathrm{L})} +$$

$$+ \int\limits_{\mathrm{P}_x} \sum_{l=1}^{r} \frac{w_l (x , y , \lambda_i) \, \omega (y , 1 + \mathrm{L})}{\omega (x , 1 + \mathrm{L})} \, \frac{\mathrm{F}_{i,l-r+1} (y , \Phi (\mathrm{P}_y))}{\omega (y , 1 + \mathrm{L})} \, \mathrm{d}\mu_y \qquad (i \in \mathrm{I}) ,$$

$$\frac{\tilde{\Phi}_{j,r} (\mathrm{P}_x)}{\omega (x , 1 + \mathrm{L})} = \frac{\Phi^0_{jr} (\mathrm{P}_x)}{\omega (x , 1 + \mathrm{L})} +$$

$$+ \int\limits_{\mathrm{P}_x} \sum_{l=1}^{r} \left\{ \frac{\mathrm{V}_l (x , y , \lambda_j) \, \omega (y , 1 + \mathrm{L})}{\omega (x , 1 + \mathrm{L})} \, \frac{\mathrm{F}_{j,l-r+1} (y , \Phi (\mathrm{P}_y))}{\omega (y , 1 + \mathrm{L})} - \right.$$

$$\left. - \frac{\mathrm{W}_l (x , y , \lambda_j) \, \omega (y , 1 + \mathrm{L})}{\omega (x , 1 + \mathrm{L})} \, \frac{\mathrm{F}_{j+1,l-r+1} (y , \Phi (\mathrm{P}_y))}{\omega (y , 1 + \mathrm{L})} \right\} \mathrm{d}\mu_y \qquad (j \in \mathrm{J}) ,$$

$$\frac{\tilde{\Phi}_{j+1,r} (\mathrm{P}_x)}{\omega (x , 1 + \mathrm{L})} = \frac{\Phi^0_{j+1,r} (\mathrm{P}_x)}{\omega (x), 1 + \mathrm{L})} +$$

$$+ \int\limits_{\mathrm{P}_x} \sum_{l=1}^{r} \left\{ \frac{\mathrm{V}_l (x , y , \lambda_j) \, \omega (y , 1 + \mathrm{L})}{\omega (x , 1 + \mathrm{L})} \, \frac{\mathrm{F}_{j+1,r-l+1} (y , \Phi (\mathrm{P}_y))}{\omega (y , 1 + \mathrm{L})} + \right.$$

$$\left. + \frac{\mathrm{W}_l (x , y , \lambda_j) \, \omega (y , 1 + \mathrm{L})}{\omega (x , 1 + \mathrm{L})} \, \frac{\mathrm{F}_{j,r-l+1} (y , \Phi (\mathrm{P}_y))}{\omega (y , 1 + \mathrm{L})} \right\} \mathrm{d}\mu_y .$$

From the properties of w_i—see § 2—we get:

$$|w_l (x , y , \lambda_i)| \leq |w_l (x , y , \mathrm{L})| \leq w_l (x , y , 1 + \mathrm{L}) ,$$

and

$$\left| \frac{w_l (x , y , \lambda_i) \, \omega (y , 1 + \mathrm{L})}{\omega (x , 1 + \mathrm{L})} \right| \leq 1 .$$

We choose now the measures h_i , p_j and q_j such that:

$$\left| \frac{\Phi^0_i (\mathrm{P}_x)}{\omega (x , 1 + \mathrm{L})} \right| \leq \tau , \tag{42}$$

τ being another, for the moment arbitrary, positive number; this choice is possible since, according to (26_1), we have

$$\left| \frac{\Phi_{ir} (\mathrm{P}_x)}{\omega (x , 1 + \mathrm{L})} \right| \leq \int\limits_{\mathrm{H}} \sum_{l=1}^{r} \left| \frac{w_l (x , y , \lambda_i) \, \omega (y , 1 + \mathrm{L})}{\omega (x , 1 + \mathrm{L})} \right| \left| \frac{\mathrm{d}h_{i,l-r+1}}{\omega (x , 1 + \mathrm{L})} \right| \leq$$

$$\leq \sum_{l=1}^{r} \int\limits_{\mathrm{H}} \left| \frac{\mathrm{d}h_{i,l-r+1}}{\omega (y , 1 + \mathrm{L})} \right| ,$$

and, according to (26_2), in the same way we obtain

$$\left|\frac{\Phi^0_{jr}(P_x)}{\omega(x, 1+L)}\right| \leq \int_H \sum_{l=1}^{r}\left(\left|\frac{dp_{j,l}}{\omega(y, 1+L)}\right| + \left|\frac{dq_{j,l}}{\omega(y, 1+L)}\right|\right).$$

If now Q is a lower bound for $\omega(x, 1+L)$, condition (42) is obviously satisfied if:

$$(43) \qquad \int_H |dh_{i,l-r+1}| \leq Q\frac{\tau}{\sigma} \quad , \quad \int_H |dp_{j,l}| \leq Q\frac{\tau}{2\sigma} \quad , \quad \int_H |dq_{j,l}| \leq Q\frac{\tau}{2\sigma},$$

where σ is the greatest multiplicity order of the eigenvalues λ.

Suppose now Φ_i, Φ_j chosen in the class of continuous functions satisfying

$$\left|\frac{\Phi_i(P_x)}{\omega(x, 1+L)}\right| \leq \eta \quad , \quad \eta < \left(\frac{1}{2\sqrt{m}\,M_2}\right)^{1/\alpha}.$$

Then, taking into account the hypothesis, (40) leads to:

$$\left|\frac{\tilde{\Phi}_{ir}(P_x)}{\omega(x, 1+L)}\right| \leq \tau + \int_{P_x} \sum_{l=1}^{r}\left|\frac{w_l(x, y, \lambda_i)\,\omega(y, 1+L)}{\omega(x, 1+L)}\right| \cdot$$

$$\cdot\left|\frac{F_{i,l-r+1}(y, \Phi(P_y))}{\omega(y, 1+L)}\right| d\mu_y \leq \tau + \sigma\int_{P_x}\frac{\tilde{K}(y)\,[\Sigma(\Phi_i(P_y))^2]^{(1+\alpha)/2}}{\omega(y, 1+L)}\, d\mu_y \leq$$

$$\leq \tau + m^{(1+\alpha)/2}\,\sigma\eta^{1+\alpha}\int_{P_x}\tilde{K}(y)\,(\omega(y, 1+L))^{\alpha}\, d\mu_y \leq$$

$$\leq \tau + \eta^{1+\alpha} M \leq \tau + \frac{1}{2\sqrt{m}\,M_2}\,\eta.$$

For the other functions, Φ_{jr}, we obtain in an analogous way the same limitation. Choose now

$$\tau < \frac{1}{2\sqrt{m}}\,\eta.$$

It follows

$$\left|\frac{\tilde{\Phi}_i(P_x)}{\omega(x, 1+L)}\right| \leq \frac{1}{2\sqrt{m}}\,\eta, \qquad \text{and} \quad \Sigma\left(\frac{\tilde{\Phi}_i(P_x)}{\omega(x, 1+L)}\right)^2 \leq \eta^2,$$

i.e., the functions $\tilde{\Phi}_k$ belong to the same class $\mathcal{M}$ as Φ_k and, as a consequence, it follows that the solution of system (38) belongs to $\mathcal{M}$; it is obvious that (39) can also be satisfied.

To come back to the stability of (9), let ε be a real positive number and choose

$$\eta < \frac{\varepsilon}{AM}.$$

[A is defined in (28) and M in (27)], and

$$\sum_{i=1}^{m} \nu_i^2 \leq \frac{a^2 Q^2}{\sigma^2 A^2 M^2} \frac{\varepsilon^2}{4}$$

[a is defined in (28) and σ and Q in (43)]. Then, taking into account (28), we have

$$\sum_{i \in I \cup J} \sum_{r=1}^{s_i} \sum_{l=1}^{r} \int_{P_x} \left| \frac{d\pi_{i,l}}{\omega(y, 1 + L)} \right|^2 \leq \frac{\sigma^2}{Q^2} \sum_{i=1}^{m} \pi_i^2 \leq$$

$$\leq \frac{\sigma^2}{a^2 Q^2} \sum_{i=1}^{m} \nu_i^2 \leq \frac{1}{A^2 M^2} \frac{\varepsilon^2}{4} \leq \frac{\eta^2}{4} .$$

It follows then

$$\sum_{i=1}^{m} \Phi_i^2 \leq M^2 \eta^2 = \frac{\varepsilon^2}{A^2} ;$$

taking again into account (28), we have

$$\sqrt{\sum_{i=1}^{m} \varphi_i^2} \leq \varepsilon ,$$

and this completes our proof.

6. Another stability theorem

The following stability theorem can be proved in conditions different from the previous ones.

THEOREM C. *Suppose that, for each i, the functions* $w_l(x, y, \lambda_i)$ $(l = 1, 2, \cdots, s_i)$ *(s_i the multiplicity of λ_i) are, in absolute value, smaller then* $\omega(x, \lambda')$, *this function satisfying the conditions*

α) $\omega(x, \lambda') > 0$;

β) *given* $\tau > 0$, *it exists* τ' *such that*

$$\|x\| > \tau' \Rightarrow \omega(x, \lambda') < \tau ,$$

the other conditions of Theorem B being realised: then, the trivial solution of (9) *is stable.*

The proof is analogous to that of Theorem B, with the only difference that $\omega(y, 1 + L)$ and $\omega(x, 1 + L)$ are substituted by $\omega(y, \lambda')$ and $\omega(x, \lambda')$ respectively.

An immediate application of this theorem is given by the case of a system of ordinary differential equations, where

$$\omega(x, y, \lambda_i) = e^{\lambda_i(x-y)} , \qquad \omega(x, \lambda_i) = e^{\lambda_i x}.$$

The stability theorem obtained is that of Liapounov, when the real parts of the eigenvalues λ_i of A are negative.

EXAMPLES: 1) Take $\Omega = \{(x) ; x_i \geq 0 , i = \overline{1 , k}\} , P_x = \{\xi_i , 0 \leq x_i \leq \xi_i , i = \overline{1 , k}\}$ and let μ be the Lebesgue measure, with weight $1/\prod_1^k (1 + x_i^2)$. The system of equations can be written as:

$$\frac{\partial^k \varphi (x)}{\partial x_1 \partial x_2 \cdots \partial x_k} = A\varphi + G (x , \varphi (P_x)) . \tag{44}$$

The functions $w (x , y , \lambda)$ and $\omega (x , \lambda)$ are in this case

$$w (x , y , \lambda) = \sum_{n=0}^{\infty} \lambda^n \frac{\prod_1^k (\operatorname{arctg} x - \operatorname{arctg} y)^n}{(n!)^k} \quad , \quad \omega (x , \lambda) = \sum_{n=0}^{\infty} \lambda^n \frac{\prod_1^k (\operatorname{arctg} x)^n}{(n!)^k} ,$$

and they are bounded. If the other conditions of Theorem B are satisfied, the trivial solution of the system (44) is stable.

2) Take as before $\Omega = \{(x_i)_i , x_i \geq 0 , i = \overline{1 , 2}\}$ and $P_x = \left\{(\xi_i) ; \xi_i \geq 0 , \sum_{i=1}^2 \xi_i^2 \leq \sum_{i=1}^2 x_i^2\right\}$. Our system is now, in polar coordinates

$$\frac{1}{r} \frac{\partial^2 \varphi}{\partial r \, \partial \theta} = A\varphi + G (x , \varphi (P_x)) \quad , \quad x = r \cos \theta \quad , \quad y = r \sin \theta .$$

Then

$$w (x , y , \lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \left(\frac{[\pi (x_1^2 + x_2^2) - \pi (y_1^2 + y_2^2)]}{4} \right)^n ,$$

$$\omega (x , \lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \left(\frac{\pi (x_1^2 + x_2^2)}{4} \right)^n ,$$

i.e.

$$w (x , y , \lambda) = \exp \left\{ \frac{\lambda\pi (x_1^2 + x_2^2) - \lambda\pi (y_1^2 + y_2^2)}{4} \right\} \quad , \quad \omega (x , \lambda) = \exp \frac{\lambda\pi (x_1^2 + x_2^2)}{4} .$$

It is obvious that these functions satisfy the hypotheses of Theorem C. However, the conclusion of this theorem cannot be applied to the system (9), since the hypotheses of the theorem on the differentiation of an integral in § 2 are not realised. Of course, the above conclusions are valid for the solution of the Volterra integral system (38), if the other hypoteses hold.

In [3] we have also given other theorems concerning the behaviour of the solution of (9).