Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Olusola Akinyele

Spectral synthesis of the algebra of zonal functions on the sphere

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **60** (1976), n.1, p. 1–5.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1976_8_60_1_1_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1976.

RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta del 10 gennaio 1976

Presiede il Presidente della Classe BENIAMINO SEGRE

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *Spectral synthesis of the algebra of zonal functions on the sphere.* Nota di OLUSOLA AKINYELE, presentata (*) dal Corrisp. G. ZAPPA.

RIASSUNTO. — Si dimostra che ogni ideale chiuso proprio nell'algebra di Banach delle funzioni zonali sulla sfera a $(k-1)$–dimensioni $(k \geq 3)$ è incluso in un ideale regolare massimale.

1. INTRODUCTION

Let G be a locally compact group, $\hat{G}$ its dual group, and $L'(G)$ the group algebra; then every proper closed ideal in $L'(G)$ is contained in a regular maximal ideal [cfr., 3]. The aim of this paper is to prove an analogue of this theorem for the algebra of zonal functions on the $(k-1)$-dimensional sphere S^{k-1} where $k \geq 3$. Our main tool lies in the theory of ultraspherical polynomials of index $\left(\frac{k}{2}-1\right)$; the so-called Gegenbauer polynomials. For the harmonic analysis on spheres we use freely the notations and results of [2].

The sphere S^{k-1} is the subset of $\mathbf{R}^k$ (k-dimensional Euclidean space) defined by $\left\{x \in \mathbf{R}^k : |x| = \left(\sum_{i=1}^{k} x_i^2\right)^{\frac{1}{2}} = 1\right\}$. Let $p = (1, 0, 0, \cdots, 0)$ be the north pole of S^{k-1} and x, y, z denote points on S^{k-1}; there is a unique finite rotation-invariant Borel measure ω on S^{k-1} such that $\omega(S^{k-1}) = 1$. We define $x \cdot y$ to be the ordinary inner product of vectors which correspond to the points x and y $(-1 \leq x \cdot y \leq 1)$. S^{k-1} admits a group of rotation which we denote by $SO(k)$ and the result of the action of the rotation $\alpha \in SO(k)$ on the point x will be denoted by $x\alpha$. The rotation operator R_α acting on functions f and

(*) Nella seduta del 13 dicembre 1975.

measures μ on S^{k-1} is defined by $R_\alpha f(x) = f(x\alpha)$ for all $x \in S^{k-1}$; $R_\alpha \mu(E) = \mu(E\alpha)$ for all μ-measurable subsets E of S^{k-1}. We shall refer to S^{k-1} simply as S and define L′ (S) as the integrable functions with respect to measure ω, such that $\int_S |f(x)| \, d\omega(x) < \infty$. Denote by M (S) the measure algebra of bounded regular Borel measure on S and the space of zonal measures by $M(S\,;p) = \{\mu \in M(S) : R_\alpha \mu = \mu$ for all $\alpha \in SO(k) \ni p\alpha = p\}$. With convolution as multiplication $M(S\,;p)$ is a Banach algebra with a unit element. The space $L'(S\,;p) = L'(S) \cap M(S\,;p)$ is a closed ideal of $M(S\,;p)$ and so it is a Banach algebra. Define similarly $C(S\,;p) = \{f \in C(S) : R_\alpha f = f$ for all $\alpha \in SO(k) \ni p\alpha = p\}$ where C (S) is the space of continuous functions on S. The space $L'(S\,;p)$ is the Banach algebra of zonal functions while $C(S\,;p)$ is a dense subspace of zonal continuous functions in $L'(S\,;p)$.

§ 2.

Let P_n^λ be the ultraspherical polynomial of degree n associated with $\lambda > 0$. A generating function for these polynomials is given by

$$(1) \qquad (1 - 2rt + r^2)^{-\lambda} = \sum_{n=0}^{\infty} \frac{\Gamma(2\lambda + n)}{n!\,\Gamma(2\lambda)} r^n P_n^\lambda(t) \,, \quad P_n^\lambda(1) = 1 \,, \quad |P_n^\lambda(t)| \leq 1 .$$

These polynomials form a complete orthogonal set on $[-1, 1]$ with respect to the measure $(1 - t^2)^{\lambda - \frac{1}{2}} dt$, and

$$(2) \qquad \int_{-1}^{1} P_n^\lambda(t) \, P_n^\lambda(t) \, (1 - t^2)^{\lambda - \frac{1}{2}} dt = \frac{\sqrt{\Pi}\,\Gamma(\lambda + \frac{1}{2})\,\Gamma(2\lambda)\,n!}{\Gamma(\lambda)\,(n + \lambda)\,\Gamma(n + 2\lambda)} .$$

It is known [2] that there exists an isometric isomorphism from the space of zonal functions on S onto $L'([-1, 1], \Omega_{(k-2)/2})$ where $\Omega_{(k-2)/2}$ is the measure given by $d\Omega(k-2)/2 = (1 - t^2)^{((k-3)/2)} dt$. In view of this isomorphism we shall assume that $\lambda = \frac{k-2}{2}$ in (1), (2) and throughout the remainder of this paper.

Suppose $f \in L'([-1, 1], \Omega_{(k-2)/2})$; then there exists a map ψ_p such that $\psi_p^{-1} f \in L'(S\,;p)$ and $\psi_p^{-1} f(x) = f(p \cdot x) = f(x_1)$. Given $f \in L'(S\,;p)$ we define the n^{th} Gegenbauer coefficients $\hat{f}_n$ of f by:

$$\hat{f}_n = \int_S f(x) \, p_n^{(k/2)-1} (p \cdot x) \, d\omega(x).$$

It is known [2] that the map $f \to \hat{f}_n$ $n = 0, 1, 2, \cdots$ are exactly the non-trivial complex-homomorphisms of $L'(S\,;p)$, and therefore the space of maximal regular ideals of $L'(S\,;p)$ is homeomorphic with the non-negative integers Z_+ with the discrete topology. If we define $SO(k-1) = \{\alpha \in SO(k) : p\alpha = p\}$, then $SO(k-1)$ is a closed subgroup of $SO(k)$ and so we can express S^{k-1} as $SO(k)/SO(k-1)$ which is a compact homogeneous space [cfr. 1, Chapt. 9]. The functions on SO (k) which are left-invariant under SO $(k-1)$ $(f(\alpha\beta)=f(\alpha)$ for all $\beta \in$ SO $(k-1))$ correspond to functions on S^{k-1} while the bi-invariant functions on SO (k) correspond to zonal functions. By essentially the same technique a generalization of the results of this paper to a compact homogenous space can be carried out. In such a generalization, the spherical harmonic functions for the compact homogeneous space play the role of the ultraspherical polynomials.

§ 3.

DEFINITION 3.1. Let A be a Banach algebra with the maximal regular ideals space $\mathcal{M}$ (A). A is said to be regular if given any closed set $F \subset \mathcal{M}$ (A) and a point $M_0 \notin F$, there exists an element $\alpha \in A$ such that $\varphi_M(\alpha)=0$ for $M \in F$, and $\varphi^{(a)}_{M_0}(\alpha) \neq 0$, where φ_M is the complex-continuous homomorphism corresponding to M.

LEMMA 3.2. $L'(S;p)$ *is a semisimple, symmetric communicative Banach algebra.*

Proof. For any $f \in L'(S;p)$, define $f^*(x)=\overline{f(x)}$ for all $x \in S$ and it is a routine metter to show that $L'(S;p)$ is symmetric. $\hat{f}_n=0$ for all $n \in Z_+$, implies $f=0$ [2] so $L'(S;p)$ is semisimple.

THEOREM 3.3. $L'(S;p)$ *is a regular Banach algebra.*

Proof. Let $F \subset Z_+$ be a closed set and assume $m \notin F$, then we can find $\psi_p^{-1} P_m^{(k/2)-1} \in C(S;p) \subset L'(S;p)$ and for $n \in F$,

$$(\psi_p^{-1} P_m^{(k/2)-1})^{\wedge}_n = \int_S \psi_p^{-1} P_m^{(k/2)-1}(x)\, P_n^{(k/2)-1}(p\cdot x)\, d\omega(x) =$$

$$= \int_S P_m^{(k/2)-1}(p\cdot x)\, P_n^{(k/2)-1}(p\cdot x)\, d\omega(x) = 0\,,$$

by the orthogonality relation of $P_n^{(k/2)-1}$ on $[-1\,,1]$.

For $x \in S$, define θ, $0 \le \theta \le \Pi$ by $p\cdot x=\cos\theta$. Using a method contained in the proof of Corollary 2.16 of [4], we can evaluate integrals over S by first integrating over the parallel $L_\theta=\{x \in S : p\cdot x=\cos\theta\}$ orthogonal to p, thereby obtaining a function of θ, $0 \le \theta \le \Pi$, which we then integrate over the interval $[0\,,\Pi]$. The measure of L_θ is

$$W_{k-2}(\sin\theta)^{k-2} = \frac{2\,\Pi^{(k-1)/2}(\sin\theta)^{k-2}}{\Gamma\left(\frac{k-1}{2}\right)}$$

where W_{k-2} is the surface area of a sphere in $\mathbf{R}^{k-2}$ of radius $\sin\theta$.

Using this fact and (2), we have

$$(\psi_p^{-1} P_m^{(k/2)-1})\hat{}_m = \int_S P_m^{(k/2)-1}(p\cdot x)\, P_m^{(k/2)-1}(p\cdot x)\, d\omega(x) =$$

$$= \frac{2\,\Pi^{(k-1)/2}}{\Gamma\left(\frac{k-1}{2}\right)} \int_0^{\pi} P_m^{(k/2)-1}(\cos\theta)\, P_m^{(k/2)-1}(\cos\theta)\,(\sin\theta)^{k-2}\, d\theta =$$

$$= \frac{2\,\Pi^{(k-1)/2}}{\Gamma\left(\frac{k-1}{2}\right)} \int_{-1}^{1} P_m^{(k/2)-1}(t)\, P_m^{(k/2)-1}(t)\,(1-t^2)^{((k-2)/2)-\frac{1}{2}}\, dt =$$

$$= \frac{2\,\Pi^{k/2}}{\sqrt{\Pi}\,\Gamma\left(\frac{k-1}{2}\right)} \; \frac{\sqrt{\Pi}\,\Gamma\left(\frac{k-2}{2}+\frac{1}{2}\right)\Gamma(k-2)\,m!}{\Gamma\left(\frac{k-2}{2}\right)\left(m+\frac{k-2}{2}\right)\Gamma(m+k-2)} =$$

$$= \frac{2\,\Pi^{k/2}\,\Gamma(k-2)\,m!}{\left(\frac{2m}{k+2}+1\right)\Gamma(m+k-2)\,\Gamma\left(\frac{k}{2}\right)}\,.$$

Since $k \geq 3$, $(\psi_p^{-1} P_m^{(k/2)-1})\hat{}_m \neq 0$, and so $L'(S\,;p)$ is regular.

Define

$$\alpha_{k,m} = \left(\frac{2\,\Pi^{k/2}\,\Gamma(k-2)\,m!}{\left(\frac{2m}{k+2}+1\right)\Gamma(m+k-2)\,\Gamma\left(\frac{k}{2}\right)}\right)$$

where $k \geq 3$ and $m = 0, 1, 2\cdots$. The following is our main theorem.

THEOREM 3.4. *Every proper closed ideal in* $L'(S\,;p)$ *is contained in a regular maximal ideal.*

Proof. Since $L'(S\,;p)$ is a regular, semisimple commutative Banach algebra, it suffices to show in view of Corollary p. 85 of [3] that the set $P = \{f \in L'(S\,;p) : \hat{f}_m$ vanishes outside a compact set in $Z_+\}$ is dense in $L'(S\,;p)$. Suppose $g \in P$, the $\exists$ a finite positive number N such that

$$\hat{g}_{N+1} = \hat{g}_{N+2} = \hat{g}_{N+3} = \cdots = 0.$$

$m \in Z_+$ and chose $\psi_p^{-1} P_m^{(k/2)-1} \in C(S\,;p)$, then using Fubini's theorem,

$$\int_S \left(g - \sum_{n=0}^{N} \alpha_{k,n}\, \hat{g}_n \psi_p^{-1} P_m^{(k/2)-1}\right)(x)\, \psi_p^{-1} P_m^{(k/2)-1}(x)\, d\omega(x) =$$

$$= \int_S g(x)\, \psi_p^{-1} P_m^{(k/2)-1}(x)\, d\omega(x)$$

$$-\sum_{n=0}^{\infty} \alpha_{k,n} \int\limits_S \int\limits_S g(x)\, \psi_p^{-1}\, P_n^{(k/2)-1}(x)\, P_n^{(k/2)-1}(p\cdot x)\, P_m^{(k/2)-1}(p\cdot x)\, d\omega(\lambda) =$$

$$= \hat{g}_m - \sum_{n=0}^{\infty} \alpha_{k,n} \int\limits_S g(x) \left\{ \int\limits_S P_n^{(k/2)-1}(p\cdot x)\, P_m^{k/2}(p\cdot x)\, d\omega(x) \right\} P_n^{(k/2)-1}(p\cdot x)\, d\omega(x) =$$

$$= \hat{g}_m - \sum_{n=0}^{\infty} \alpha_{k,n} \int\limits_S g(x) \frac{\delta_{mn}}{\alpha_{k,m}} P_n^{(k/2)-1}(p\cdot x)\, d\omega(x) =$$

$$= 0 .$$

Since $\{\psi_p^{-1} P_n^{(k/2)-1} ; n = 0, 1, 2, \cdots\}$ is dense in $C(S; p)$, it follows that

$$\int\limits_S \left(g - \sum_{n=0}^{N} \alpha_{k,n} \hat{g}_n \Psi_p^{-1} P_n^{(k/2)-1} \right)(x) f(x)\, d\omega(x) = 0$$

for all $f \in C(S; p)$ which implies that $g = \sum_{n-0}^{N} \alpha_{k,n} \hat{g}_n \psi_p^{-1} P_n^{(k/2)-1}$. Since finite linear combinations of $\{\psi_p^{-1} P_n^{(k/2)-1}, n \in Z_+\}$ are dense in $C(S; p)$, then the set P is dense in $L'(S; p)$ and the proof is complete.

REFERENCES

[1] C. F. DUNKL and D. E. RAMIREZ (1971) – *Topics in Harmonic Analysis*, Appleton-Century-Crofts, New York.

[2] C. F. DUNKL (1966) – *Operators and harmonic analyis on the sphere*, «Trans. Amer. Math. Soc.», *125*, 250–263.

[3] L. H. LOONUS (1953) – *An Introduction to Abstract Harmonic Analysis*, Van Nostrand, New York.

[4] E. M. STEIN and G. WEISS (1971) – *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press.