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Asymptotic behaviour of the electric field near the singular points of the conductor surface

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Analisi matematica. — *Asymptotic behaviour of the electric field near the singular points of the conductor surface* (*). Nota (**) del Corrisp. GAETANO FICHERA.

RIASSUNTO. — Vengono dati risultati che descrivono analiticamente il cosiddetto « effetto delle punte » di un conduttore elettrico.

Let A be a bounded domain (connected open set) of the n-dimensional cartesian space X^n $(n \geq 2)$. Suppose that the origin o of X^n is contained in ∂A. Let S^{n-1} be the unit sphere $|x| = 1$ of X^n. Denote by Ω a domain of S^{n-1}, which does not coincide with S^{n-1}. Let H be the cone obtained through projection of $\overline{\Omega}$ from o, i.e. we denote by H the point set of X^n

$$H \equiv \left\{ x \,;\, 0 \leq |x| < 1 \,,\, \omega = \frac{x}{|x|} \in \overline{\Omega} \quad \text{for} \quad x \neq o \right\}.$$

If we denote by B^n the unit ball $|x| < 1$, we suppose that $\bar{A} \cap B^n = H$.

Let u be a harmonic function belonging to $\mathcal{H}_1(A)$ and φ a function of $C^\infty(X^n)$. Suppose that $u - \varphi \in \overset{\circ}{\mathcal{H}}_1(A)$. If we assume that $\partial\Omega$ is sufficiently smooth, then $u \in C^0(H)$.

The problem now arises: under what conditions on φ we have $u \in C^1(H)$?

Let us first suppose that $\partial\Omega$ *is* C^∞*-smooth* (see [3], p. 52).

Set $u - \varphi = U$, $-\Delta_2 \varphi = \Phi$. An equivalent statement of our problem is the following: find under what conditions on Φ we have $U \in C^1(H)$.

Let R be an arbitrary positive number less than 1. Assume $0 < R_1 < R_2 < R$ and denote by $\psi(x)$ a *cut off function*, i.e. a function belonging to $C^\infty(X^n)$ which coincides with 1 for $|x| < R_1$ and vanishes for $|x| > R_2$. Set $w = \psi U$. For the arbitrariness of R, R_1, R_2 our problem consists in finding the conditions for Φ under which $u \in C^1(H_R)$, where H_R is the intersection of H with the domain $|x| < R$.

Let L_ω be the Laplace-Beltrami operator on S^{n-1} and consider the classical eigenvalue problem in Ω

$$(1) \qquad L_\omega v + \lambda v = 0 \quad \text{on} \quad \Omega\,, \qquad v|_{\partial\Omega} = 0\,.$$

This problem has an increasing sequence of positive eigenvalues $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ Let $\{v_h\}$ be a corresponding orthonormal complete system of eigenfunctions.

(*) Results of this paper will be announced in a lecture at the Conference on Partial Differential Equations to be held in Moscow (26 January 1976, 31 January 1976) to honour the 75th Anniversary of I. G. Petrowski (1901–1973).

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Set for $x \neq 0$, $\omega = x \, | x |^{-1}$, $\rho = | x |$, and for $h = 1, 2, \cdots$

$$w_h(\rho) = \int_{\Omega} w(\rho\omega) \, v_h(\omega) \, d\omega ,$$

where $d\omega$ is the unit hypersurface measure on S^{n-1}.

If we set

$$\Delta_2 w = f , \tag{2}$$

$$f_h(\rho) = \int_{\Omega} f(\rho\omega) \, v_h(\omega) \, d\omega ,$$

from (2) we deduce by multiplication for $v_h(\omega)$ and after integration over Ω:

$$w_h''(\rho) + \frac{n-1}{\rho} w_h'(\rho) + \frac{1}{\rho^2} \int_{\Omega} v_h(\omega) \, L_\omega w(\rho\omega) \, d\omega = f_h(\rho) .$$

Hence, from (1),

$$w_h''(\rho) + \frac{n-1}{\rho} w_h'(\rho) - \frac{\lambda_h}{\rho^2} w_h(\rho) = f_h(\rho) . \tag{3}$$

Set

$$\alpha_h = \frac{\sqrt{(n-2)^2 + 4\lambda_h} - n + 2}{2} , \qquad \beta_h = \frac{\sqrt{(n-2)^2 + 4\lambda_h} + n - 2}{2} . \tag{4}$$

Since $w(R) = w'(R) = 0$, from (3) we deduce

$$(\alpha_h + \beta_h) \, w_h(\rho) = \int_R^{\rho} \left(\frac{\rho^{\alpha_h}}{r^{\alpha_h}} - \frac{r^{\beta_h}}{\rho^{\beta_h}} \right) f_h(r) \, r dr . \tag{5}$$

The function $w_h(\rho)$ is continuous for $\rho \to 0^+$, hence we must have

$$\int_0^R r^{\beta_h + 1} f_h(r) \, dr = 0 .$$

That enables us to write Eq. (5) as follows

$$-(\alpha_h + \beta_h) \, w_h(\rho) = \rho^{-\beta_h} \int_0^{\rho} f_h(r) \, r^{\beta_h + 1} \, dr + \rho^{\alpha_h} \int_{\rho}^{R} f_h(r) \, r^{-\alpha_h + 1} \, dr .$$

We deduce for $0 < \rho < R$

$$-(\alpha_h + \beta_h) \, w_h'(\rho) = -\beta_h \, \rho^{-\beta_h - 1} \int_0^{\rho} f_h(r) \, r^{\beta_h + 1} \, dr + \alpha_h \, \rho^{\alpha_h - 1} \int_{\rho}^{R} f_h(r) \, r^{-\alpha_h + 1} \, dr .$$

If $w \in C^1(H_R)$ the function $w_h'(\rho)$ is continuous for $\rho \to 0^+$. If M is such that $|f| \leq M$ in A, we have

$$\left| \frac{1}{\rho^{\beta_h+1}} \int_0^\rho f_h(r)\, r^{\beta_h+1}\, dr \right| \leq \frac{1}{\beta_h + 2} M\rho .$$

The function

$$\int_\rho^R f_h(r) \left(\frac{\rho}{r}\right)^{\alpha_h - 1} dr \tag{6}$$

is continuous for $\rho \to 0^+$ if $\alpha_h \geq 1$. If $\alpha_h < 1$ the function (6) is continuous for $\rho \to 0^+$ if and only if

$$\int_0^R f_h(r)\, r^{-\alpha_h+1}\, dr = 0 . \tag{7}$$

Hence a necessary condition for $u \in C^1(H)$ is that (7) holds for every h such that $\alpha_h < 1$, i.e. $\lambda_h < n - 1$.

It is not difficult to prove that conditions (7) are sufficient for $u \in C^1(H)$. This can be done by using the development of $w(\rho\omega)$

$$w(\rho\omega) = \sum_{k=1}^{\infty} w_h(\rho)\, v_h(\omega)\ ^{(1)} .$$

Set $V_h(x) = \rho^{-\alpha_h - n + 2}\, v_h(\omega)\, \psi(x)$. We may suppose that $V_h(x)$ is defined in A, by assuming $V_h(x) = 0$ for $|x| > R_2$. For every $q \in L^2(A)$ denote by $z = Gq$ the variational solution of the Dirichlet problem

$$\Delta_2 z = q \qquad \text{in } A , \qquad z|_{\partial A} = 0 .$$

The following theorem holds.

I. *Under the above hypothesis on* $\partial\Omega$, *necessary and sufficient condition for* $u \in C^1(H)$ *is that*

$$\int_A (V_h - G\Delta_2 V_h)\, \Phi\, dx = 0 \tag{8}$$

for every h *such that* $\lambda_h < n - 1$.

(1) The technique to be used is the one largely employed in the paper [4].

We have only to prove that Eq. (7) is equivalent to Eq. (8). We have, setting $\overset{\circ}{\mathrm{V}}_h(x) = \rho^{-\alpha_h - n + 2} v_h(\omega)$,

$$\int_0^{\mathrm{R}} f_h(r)\, r^{-\alpha_h+1}\, \mathrm{d}r = \lim_{\varepsilon\to 0+} \int_\varepsilon^{\mathrm{R}} r^{n-1}\, \mathrm{d}r \int_\Omega r^{-\alpha_h-n+2} v_h(\omega) f(\rho\,,\omega)\, \mathrm{d}\omega =$$

$$= \lim_{\varepsilon\to 0+} \int_{\mathrm{A}-\mathrm{H}_\varepsilon} \mathrm{V}_h\, \Phi\, \mathrm{d}x + 2\int_{\mathrm{A}} \overset{\circ}{\mathrm{V}}_h \operatorname{grad} \mathrm{U} \operatorname{grad} \psi \mathrm{d}x + \int_{\mathrm{A}} \overset{\circ}{\mathrm{V}}_h \mathrm{U}\Delta_2\, \psi \mathrm{d}x$$

$$= \int_{\mathrm{A}} \mathrm{V}_h\, \Phi\, \mathrm{d}x - 2\int_{\mathrm{A}} \mathrm{U} \operatorname{grad} \overset{\circ}{\mathrm{V}}_h \operatorname{grad} \psi\, \mathrm{d}x - \int_{\mathrm{A}} \mathrm{U}\overset{\circ}{\mathrm{V}}_h\, \Delta_2\, \psi\, \mathrm{d}x$$

$$= \int_{\mathrm{A}} \mathrm{V}_h\, \Phi\, \mathrm{d}x - \int_{\mathrm{A}} \mathrm{U}\Delta_2\, \mathrm{V}_h\, \mathrm{d}x = \int_{\mathrm{A}} (\mathrm{V}_h - \mathrm{G}\Delta_2\, \mathrm{V}_h)\, \Phi\, \mathrm{d}x\,.$$

It must be remarked that the function $\mathrm{V}_h - \mathrm{G}\Delta_2 \mathrm{V}_h$ does not depend on the cut off function ψ; that is easily seen.

The compatibility conditions (8) occur when and only when for some $h : \lambda_h < n - 1$; hence, if there are any of such compatibility conditions, they are a finite set. If H is convex, since in this case $\lambda_1 > n - 1$, we have $u \in \mathrm{C}^1(\mathrm{H})$ for every $\varphi \in \mathrm{C}^\infty(\mathrm{X}^n)$.

It is not difficult to extend the preceeding analysis and find the conditions for $u \in \mathrm{C}^m(\mathrm{H})$ for any given $m \geq 1$.

The situation is—in general—very different from Theorem I, if $\partial\Omega$ is not smooth. Since the case $n = 2$ is covered by Theorem I, we shall perform our analysis in the simplest case, i.e. $n = 3$.

Let us assume that $\partial\Omega$ is a connected set formed by a finite collection of simple C^∞ arcs, two of them meeting eventually only in one end point. Let $\omega_1, \cdots, \omega_q$ be the vertices of $\partial\Omega$ and let μ_k be the size of the angle of $\partial\Omega$ in ω_k (μ_k is measured from the interior of Ω; $0 \leq \mu_k \leq 2\pi$).

Let us denote by I_k the segment of X^3 defined by the condition

$$0 \leq |x| < 1\,, \qquad x = |x|\, \omega_k\,.$$

By using techniques developed in [4], the following theorem can be proved.

II. *For* $x \in \mathrm{H} - (\mathrm{I}_1 \cup \cdots \cup \mathrm{I}_q)\,,\ |x| < \mathrm{R}$, *we have*

$$|\,[\operatorname{grad}(u - \varphi)]_x\,| = \mathrm{O}\left[\sigma(\rho\,,\alpha) \prod_{k=1}^{q} \tau_k(\omega)\right] \tag{9}$$

where $\alpha = \alpha_1$ [see (4)], $\rho = |x|$, $\omega = x\,|x|^{-1}$,

$$\sigma(\rho\,,\alpha) \begin{cases} = \rho^{\alpha-1} & 0 < \alpha < 2, \\ = \rho \log \dfrac{\mathrm{d}}{\rho} & \alpha = 2, \\ = \rho & \alpha > 2, \end{cases}$$

$$\tau_k(\omega) \begin{cases} = |\omega - \omega_k|^{\frac{\pi}{\mu_k}-1} & \mu_k > \frac{\pi}{2}, \\ = |\omega - \omega_k| \log \dfrac{d}{|\omega - \omega_k|} & \mu_k = \frac{\pi}{2}, \\ = |\omega - \omega_k| & \mu_k < \frac{\pi}{2}. \end{cases}$$

d *is the diameter of* H.

The asymptotic estimate (9) *cannot be improved.*

The meaning of the last statement of the theorem is fully explained in [4].

This theorem points out that the problem of the behaviour of the electric field near the singularities of the conductor surface is essentially a problem of refined Numerical Analysis: *find close lower and upper bounds for* α, *i.e. for the first eigenvalue* λ_1 *of problem* (1).

This problem is, in general, extremely difficult.

Theorem II shows that if $\partial\Omega$ has singularities, the singularities of grad u can occur not only in the vertex o but also along the edges corresponding to every ω_k such that $\mu_k > \pi$.

However from Theorem II we deduce (see [4]):

III. *We have* $u \in C^1(H)$ *for every* $\varphi \in C^\infty(X^3)$ *when and only when* $\mu_k \leq \pi$ $(k = 1, \cdots, q)$ *and* $\alpha \geq 1$.

The following theorems are interesting for applications, especially in Electrostatics; they refer to the case that $\alpha < 1$ or $\mu_k > \pi$ for some k.

IV. *Let* $\bar{p}$ *be the smallest of the positive numbers of the following set*:

$$\frac{3}{1-\alpha}, \frac{2\mu_1}{\mu_1 - \pi}, \cdots, \frac{2\mu_q}{\mu_q - \pi}.$$

We have $|\text{grad}\, u| \in L^p(H_R)$ *for* $1 \leq p < \bar{p}$.

The statement $|\text{grad}\, u| \in L^{p'}(H_R)$ *with* $p' \geq \bar{p}$ *is, in general, false.*

V. *Let* $\bar{p}$ *be the smallest of the positive numbers of the following set*:

$$\frac{2}{1-\alpha}, \frac{\mu_1}{\mu_1 - \pi}, \cdots, \frac{\mu_q}{\mu_q - \pi}.$$

We have $\dfrac{\partial u}{\partial \nu} \in L^p(\partial_1 H_R)$ $\left(\dfrac{\partial}{\partial \nu}\right.$ *denotes normal differentiation in any regular point of* $\left.\partial_1 H_R = \partial H \cap B_R\,;\, B_R : |x| < R\right)$ *for* $1 \leq p < \bar{p}$.

The statement $\dfrac{\partial u}{\partial \nu} \in L^{p'}(\partial_1 H_R)$ *with* $p' \geq \bar{p}$ *is, in general, false.*

Let us remark that we have $\bar{p} > 3$ in the Theorem IV and $\bar{p} \geq 2$ in the Theorem V. In [4], [5] the main problem, consisting in giving lower and upper bounds for α, has been solved in the case that A is the exterior of the cubic domain: $0 < x_1 < 1$, $0 < x_2 < 1$, $0 < x_3 < 1$ [(2)]. The following bounds for α have been obtained:

$$0.4335 < \alpha < 0.4645.$$

Theorems IV, V and the lower bound obtained for α permit to state that (assuming without any loss that grad φ has a bounded support): $|\operatorname{grad} u| \in L^{\frac{6000}{1133}}(A)$ and $\frac{\partial u}{\partial \nu} \in L^{3-\varepsilon}(\partial A)$ (for any $\varepsilon : 0 < \varepsilon \leq 2$).

Returning to the conditions for $u \in C^1(H)$, it turns out that the problem is now more complicated. In fact suppose that $\mu_k > \pi$. The conditions for the continuous differentiability of u near any point x^0 of I_k distinct from o, cannot be expressed by a finite set of integral conditions on Φ.

For proving this statement consider in the X^3 space a cylindrical coordinate system (ρ, θ, z)

$$x_1 = \rho \cos \theta \quad , \quad x_2 = \rho \sin \theta \quad , \quad x_3 = z \quad (\rho \geq 0 , 0 \leq \theta < 2\pi) .$$

Let K be the domain (dihedral angle) defined by the conditions

$$0 < \rho < 1 \quad , \quad 0 < \theta < \mu .$$

Let A be a bounded domain of X^3 such that for $0 < R < 1 : \bar{A} \cap \bar{B}_R = \bar{K} \cap \bar{B}_R$.

Suppose that u is a harmonic function belonging to $\mathcal{H}_1(A)$ and such that $u - \varphi \in \mathring{\mathcal{H}}_1(A)$.

We consider the problem consisting in finding the conditions to be imposed on φ for being $u \in C^1(\bar{K} \cap \bar{B}_R)$.

If we have $\mu \leq \pi$, then $u \in C^1(\bar{K} \cap \bar{B}_R)$ (see [4]). Suppose $\mu > \pi$. Set for $h = 1, 2, \cdots$

$$U_h(x) = \psi(x)\, I_{-(\pi/\mu)}\left(\frac{h\pi}{\mu}\rho\right) \sin \frac{\pi}{\mu}\theta \sin h\pi z ,$$

where $\psi(x)$ is the above introduced cut off function and $I_{-(\pi/\mu)}(t)$ is the modified Bessel function of the first kind (see [14], p. 372).

(2) In this case A is not bounded and the further condition $u(\infty) = 0$ must be imposed on u.

By a proof similar to the proof of Theorem I, we get the following:

VI. *Necessary and sufficient conditions for* $u \in C^1(\overline{K} \cap \overline{B}_R)$ *are the following*:

$$\int_A (U_h - G\Delta_2 U_h)\, \Phi \, dx = 0 \qquad (h = 1, 2, \cdots).$$

G has the above introduced meaning.

(Remark that, for $\mu = 2\pi$, $C^1(\overline{K} \cap \overline{B}_R)$ has a self-explanatory meaning different from $C^1(\overline{B}_R)$).

Different from Theorem I, we have in this case a countable set of compatibility conditions.

Thus in the case that $\partial\Omega$ has some vertices and $\mu_k > \pi$ for some k, the conditions for $u \in C^1(H)$ cannot be expressed by a finite set of integral conditions on Φ. While the conditions obtained by combining Theorem I and Theorem VI are necessary for $u \in C^1(H)$, it is not known to the writer whether the set of all these conditions (which, obviously, can be reduced to a countable set) are sufficient for $u \in C^1(H)$.

Short bibliographical remarks. There is an extensive bibliography dealing with elliptic differential equations in a domain with a singular boundary. The case $n = 2$, although very important for several applications, is less interesting from a theoretical point of view and easier to be handled. On the other hand for $n = 2$ the theory of analytic functions of one complex variable is of great help. For $n \geq 3$ the investigations in this field were started by a celebrated paper by Carleman [1]. It is impossible to refer here to all the papers devoted to this subject. However we must quote the outstanding papers by Kondrat'ev [7], [8], [9] and by Hanna and Smith [6], which have been the starting points of modern investigations in this field. Soviet Mathematicians have been very active in this field. Let us quote here the recent work of V. G. Maz'ja and B. A. Plamenevskii [10], [11], [12] who deal with a very general class of problems. The papers by M. A. Sneider [13] and P. Castellani Rizzonelli [2] solve important concrete problems for Electrostatics and Elasticity, respectively, in a domain with a singular boundary.

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