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Fisica matematica. — *Maxwell's equations and Clifford algebra: space-time formulation* (*). Nota II di AMALIA ERCOLI FINZI (**) e CARLO MOROSI (**), presentata (***) dal Socio C. CATTANEO.

RIASSUNTO. — Si analizza la formulazione spazio–temporale delle equazioni generalizzate di campo elettromagnetico, ottenute in [1] secondo il formalismo dell'algebra di Clifford. Da tali equazioni si risale alla formulazione variazionale e alla descrizione in termini di un tensore sforzi–energia, senza che l'estensione del formalismo classico implichi, prescindendo da dati sperimentali, una riduzione del campo generalizzato al campo classico.

INTRODUCTION

In this paper the previous analysis is continued and completed of the properties of a generalized formulation, by means of the formalism of the Pauli algebra, of the Maxwell's equations for the electromagnetic field [1]: the generalized field equations are as follows (1):

$$\text{(a)}\quad \begin{cases} \dfrac{\partial\alpha}{\partial t} + \operatorname{div}\mathbf{E} = \rho \quad ; & \dfrac{\partial\beta}{\partial t} + \operatorname{div}\mathbf{B} = \sigma \\[2mm] \dfrac{\partial\mathbf{E}}{\partial t} - \operatorname{rot}\mathbf{B} + \operatorname{grad}\alpha = -\boldsymbol{j} \quad ; & \dfrac{\partial\mathbf{B}}{\partial t} + \operatorname{rot}\mathbf{E} + \operatorname{grad}\beta = -\boldsymbol{m}\,; \end{cases}$$

they give the time evolution of the classic field $(\mathbf{E}, \mathbf{B})$ and of a new scalar field (α, β).

In the following three sections the possibility is examined to give to eqs. (*a*) and to those derived from them an invariant tensor form, for a flat space-time four-manifold: the conclusion can be drawn that the generalized equations can actually be given in a tensor form (moreover, different but equivalent formulations are possible), without particular restrictions upon the generalized field. Starting from one of these formulations a variational principle can be constructed similar to a classic principle (Belinfante-Infeld).

By means of another formulation, it is easy to give a description in terms of the same stress-energy tensor obtained in [1]; this tensor can be made symmetric: in this form, it maintains the properties of the classic tensor, being expressed in terms of a positive definite generalized energy, a generalized Poynting vector and a generalized stress.

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(1) Eqs. (*a*) are written for the vacuum, with $\varepsilon_0 = \mu_0 = c = 1$: see [1] for the discussion of these equations and a possible interpretation of $\alpha, \beta, \sigma, \boldsymbol{m}$.

Thus the conclusions of [1] are confirmed by the results of this paper: the classic electromagnetic field can be generalized in such a way to maintain any formal property of the Maxwell field; it reduces to the classic field only by taking into account the experimental evidence that no magnetic charged current density exists and that the electric charge is conserved: the critical feature of this fact is thus stressed.

1. Space-time formulation

In this section a tensor form [2] is given to the generalized Maxwell's equations (*a*): to this end we define the complex four-vectors

$$j_\nu \equiv (\rho + i\sigma\,;\, -\boldsymbol{j} - i\boldsymbol{m}) \quad ; \quad A_\nu \equiv (\varphi + i\psi\,;\, -\mathbf{A} - i\mathbf{C}) \tag{1.1}$$

and the tensor

$$H_{\mu\nu} \equiv \begin{bmatrix} \alpha + i\beta & E_x + iB_x & E_y + iB_y & E_z + iB_z \\ -(E_x + iB_x) & -(\alpha + i\beta) & -B_z + iE_z & B_y - iE_y \\ -(E_y + iB_y) & B_z - iE_z & -(\alpha + i\beta) & -B_x + iE_x \\ -(E_z + iB_z) & -B_y + iE_y & B_x - iE_x & -(\alpha + i\beta) \end{bmatrix} . \tag{1.2}$$

$H_{\mu\nu}$ can be expressed as follows:

$$H_{\mu\nu} \equiv g_{\mu\nu}(\alpha + i\beta) + F_{\mu\nu} - \frac{i}{2}\,\varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} , \tag{1.3}$$

where $F_{\mu\nu}$ is the well-known Maxwell electromagnetic tensor.

By the above notation, the first order field equations take the form (as we will see, this formulation is not unique)

$$H_{\mu\nu}{}^{/\mu} = j_\nu \tag{1.4}$$

and the equations of the second order are given by

$$\square H_{\mu\nu} \equiv g^{\rho\sigma} \nabla_\rho \nabla_\sigma H_{\mu\nu} = j_{\mu/\nu} - j_{\nu/\mu} + g_{\mu\nu}\, j^{\sigma}{}_{/\sigma} + i\varepsilon_{\mu\nu\rho\sigma}\, j^{\rho/\sigma} . \tag{1.5}$$

We remark that eq. (1.4) corresponds to all the four eqs. (*a*), whereas the analogue classic equation, $F_{\alpha\beta}{}^{/\alpha} = j_\beta$, corresponds only to one group of equations. The generalized field can be expressed in terms of the four-potential,

(2) The equations are therefore written in a form valid for any choice of coordinates, and the usual notations of tensor calculus are used. The Cartesian form of the equations is immediately obtained if the Minkowsky metric tensor is chosen as $\eta_{\mu\nu} = \mathrm{diag}(1\,; -1\,; -1\,; -1)$, $\varepsilon_{\mu\nu\rho\sigma}$ becoming the permutation symbol. We remember that the (covariant) second order derivatives commute, the space-time manifold being flat: $\nabla_\mu \nabla_\nu \equiv /_{\mu\nu} = /_{\nu\mu} \equiv \nabla_\nu \nabla_\mu$.

and the potential in terms of the current, by the following equations, respectively:

$$H_{\mu\nu} = A_{\nu/\mu} - A_{\mu/\nu} + g_{\mu\nu} A^{\lambda}{}_{/\lambda} + i\varepsilon_{\mu\nu\rho\sigma} A^{\rho/\sigma} \tag{1.6}$$

$$\Box A_\nu = j_\nu , \tag{1.7}$$

where eq. (1.7) is formally the same as the classic one.

As for the relation between this formulation and the classic equations, we remark that eq. (1.4) is equivalent to the classic formulation (by separating the imaginary and real parts) if $\alpha , \beta , \sigma , \boldsymbol{m}$ vanish: if ψ and $\mathbf{C}$ vanish, the same is true for eq. (1.6). As $A^{\nu}{}_{/\nu} = (\alpha + i\beta)$ by eq. (1.6), the Lorentz gauge is implied when the formulation reduces to the classic one by putting $\alpha = \beta = 0$.

As can be easily verified, eq. (1.6), for $\alpha \neq 0 , \beta \neq 0$, is invariant only under restricted gauge transformations of the second kind, for the unique gauge transformation $A_\nu = B_\nu + u_\nu$, under which $H_{\mu\nu}(A) = H_{\mu\nu}(B)$, is the following:

$$u_\nu = \lambda_{/\nu} + i\mu_{/\nu} , \qquad \text{with} \quad \Box\lambda = \Box\mu = 0 . \tag{1.8}$$

Therefore, owing to the skew-symmetry: of $H_{\mu\nu} \to F_{\mu\nu}$, $A^{\sigma}{}_{/\sigma}$ must vanish in the reduction to the classic formulation, whereas in the present generalized formulation the Lorentz condition is no more required, and $A^{\sigma}{}_{/\sigma}$ is invariant under the gauge transformation (1.8). At last let us compute the divergence $j^{\sigma}{}_{/\sigma}$ of the current four-vector, wihch is given, by eqs. (1.4) and (1.6), by:

$$j^{\sigma}{}_{/\sigma} = H^{\nu\sigma}{}_{/\nu\sigma} = \Box(\alpha + i\beta) . \tag{1.9}$$

As already stressed [1], the charge is no more conserved, owing to the existence of the " scalar " field (α , β): on the other hand the condition $j^{\sigma}{}_{/\sigma} = 0$ implies the vanishing of the scalar field, under the hypothesis of uniqueness of the field for suitable " boundary " conditions. Thus it seems possible to describe completely the generalized field by means of a complex tensor of the second order, $H_{\mu\nu}$, with eigth different components, or by means of a skew-symmetric tensor, $F_{\mu\nu}$, and an isotropic tensor, $g_{\mu\nu}(\alpha + i\beta)$ (3).

2. VARIATIONAL FORMULATION

Eq. (1.4) does not seem to be directly derivable from a variational principle, but it can be replaced by an equivalent form, i.e.

$$H_{\nu\mu}{}^{/\mu} - H_{\mu\nu}{}^{/\mu} - H^{\sigma}{}_{\cdot\sigma/\nu} - i\varepsilon_{\nu\mu\rho\sigma} H^{\mu\rho/\sigma} = -4 j_\nu . \tag{2.1}$$

(3) Among other generalizations of the electromagnetic field, see [2], where the field is expressed in terms of a skew-symmetric tensor of the second order, a scalar field and a completely skew-symmetric tensor of the fourth order.

Now, if eqs. (1.6) and (2.1) are taken into account, that is both the generalized field equations and the relation between the field and the potential, both of them can be derived by the stationarity of the same functional (4). Eq. (1.6) can be written in operator form

$$\mathcal{D}a - h = 0\,, \tag{2.2}$$

where the following correspondences are made:

$$\begin{cases} H_{\mu\nu} \to h \quad ; \quad A_\nu \to a\,; \\ A_{\mu/\nu} - A_{\nu/\mu} + g_{\mu\nu}\, A^{\sigma}{}_{/\sigma} + i\varepsilon_{\nu\mu\rho\sigma}\, A^{\rho/\sigma} \equiv [g_{\mu\sigma}\,\nabla_\nu - g_{\nu\sigma}\,\nabla_\mu + \\ \qquad + g_{\mu\nu}\,\nabla_\sigma + i g^{\rho\lambda}\,\varepsilon_{\nu\mu\sigma\rho}\,\nabla_\lambda]\, A^{\sigma} \equiv \mathcal{D}_{\mu\nu\sigma}\, A^{\sigma} \to \mathcal{D}a. \end{cases} \tag{2.3}$$

In an analogous way, for the field equations one has:

$$\mathcal{C}h = -4j\,, \tag{2.4}$$

where

$$\begin{aligned} &- H_{\lambda\sigma}{}^{/\lambda} + H_{\sigma\lambda}{}^{/\lambda} - H^{\lambda}_{\cdot\lambda/\sigma} - i\varepsilon_{\sigma\lambda\rho\delta}\, H^{\lambda\rho/\sigma} \equiv [-g_{\sigma\nu}\,\nabla_\mu + g_{\sigma\mu}\,\Delta_\nu - \\ &\quad - g_{\mu\nu}\,\nabla_\sigma - i\varepsilon_{\sigma\mu\nu\cdot}{}^{\delta}\,\nabla_\delta]\, H^{\mu\nu} \equiv \mathcal{C}_{\sigma\mu\nu}\, H^{\mu\nu} \to \mathcal{C}h \quad ; \quad j_\nu \to j\,. \end{aligned} \tag{2.5}$$

Both equations (2.2) and (2.4) are collected into the operator equation:

$$\mathcal{N}u = v\,, \tag{2.6}$$

where:

$$\mathcal{N} \equiv \begin{pmatrix} 0 & \mathcal{C} \\ \mathcal{D} & -\mathrm{I} \end{pmatrix} \quad ; \quad u \equiv \begin{pmatrix} a \\ h \end{pmatrix} \quad ; \quad v \equiv \begin{pmatrix} -4j \\ 0 \end{pmatrix}. \tag{2.7}$$

As well-known [3, ch. 2], eq. (2.6) has a variational formulation if the Geteaux derivative $\mathcal{N}_u'$ of $\mathcal{N}$ is symmetric, that is if:

$$\tilde{\mathcal{C}}_h' = \mathcal{D}_a' \Rightarrow \tilde{\mathcal{C}} = \mathcal{D}\,, \tag{2.8}$$

where $\mathcal{C}$ and $\mathcal{D}$ are defined by (2.3) and (2.5). The upper condition depends upon the linearity of the operators just defined, for the Gateaux derivative of a linear operator is the operator itself (as mentioned above, the boundary conditions of the domain are not considered: if the domains were taken into

(4) Here and in what follows, we consider only a "formal" variational formulation, i.e. we do not take into account initial, final and boundary conditions which have to be associated with the equations in order to obtain a complete formulation of the problem and its variational derivation. This we have only formal operators, written by capital Italic letters.

account, the generalization to non-linear operators would be straightforward): thus condition (2.8) implies that

$$(2.9)\qquad \langle \mathscr{D}a\,,h\rangle - \langle a\,,\mathscr{C}h\rangle = \int_\Omega [A_{\mu/\nu} - A_{\nu/\mu} + g_{\nu\mu}\,A^{\lambda}{}_{/\lambda} + i\varepsilon_{\nu\mu\gamma\delta}\,A^{\gamma/\delta}]\,H^{\nu\mu}\,d\Omega -$$

$$- \int_\Omega A^{\sigma}\,[-H_{\lambda\sigma}{}^{/\lambda} + H_{\sigma\lambda}{}^{/\lambda} - H^{\lambda}{}_{\cdot\lambda/\sigma} - i\varepsilon_{\sigma\lambda\gamma\delta}\,H^{\lambda\gamma/\delta}]\,d\Omega = 0$$

where $d\Omega = \sqrt{g}\,dx = \sqrt{g}\,dx^0\,dx^1\,dx^2\,dx^3$, $g = |\det g_{\mu\nu}|$.

By trivial integrations by parts, eq. (2.9) is immediately verified, hence problem (2.6) can be deduced from the stationarity of the functional:

$$(2.10)\qquad \mathscr{F}[u] = \tfrac{1}{2}\langle u\,,\mathscr{N}u\rangle - \langle u\,,v\rangle\,,$$

whose explicit form is as follows:

$$(2.11)\qquad \mathscr{F}(h\,,a) = \tfrac{1}{2}\langle a\,,\mathscr{C}h\rangle + \tfrac{1}{2}\langle h\,,\mathscr{D}a\rangle - \tfrac{1}{2}\langle h\,,h\rangle + 4\langle a\,,j\rangle =$$

$$= \int_\Omega [H_{\mu\nu}\,(A^{\nu/\mu} - A^{\mu/\nu} + g^{\mu\nu}\,A^{\lambda}{}_{/\lambda} + i\varepsilon^{\mu\nu\lambda\delta}\,A_{\lambda/\delta} - \tfrac{1}{2}\,H^{\mu\nu}) + 4A^{\lambda}\,j_{\lambda}]\,d\Omega\,.$$

At last, taking into account the Minkowski metric and definitions (1.1) and (1.2), one has:

$$(2.12)\qquad \mathscr{F} = \int_\Omega (U + iV)\,dx = \int_\Omega \mathscr{L}\,dx$$

where:

$$(2.13)\qquad \begin{cases} U = 4\,\mathbf{E}\cdot\left(\operatorname{grad}\varphi + \dfrac{\partial\mathbf{A}}{\partial t} + \operatorname{rot}\mathbf{C}\right) - 4\,\mathbf{B}\cdot\left(\operatorname{grad}\psi + \dfrac{\partial\mathbf{C}}{\partial t} - \operatorname{rot}\mathbf{A}\right) + \\[2mm] \quad + 4\alpha\left(\dfrac{\partial\varphi}{\partial t} + \operatorname{div}\mathbf{A}\right) - 4\beta\left(\dfrac{\partial\psi}{\partial t} + \operatorname{div}\mathbf{C}\right) - 2\,(\alpha^2 - \beta^2 + E^2 - B^2) + \\[2mm] \quad + 4\,(\rho\varphi - \sigma\psi - \mathbf{A}\cdot\boldsymbol{j} + \mathbf{C}\cdot\boldsymbol{m}) \\[2mm] V = 4\,\mathbf{E}\cdot\left(\operatorname{grad}\psi + \dfrac{\partial\mathbf{C}}{\partial t} - \operatorname{rot}\mathbf{A}\right) + 4\,\mathbf{B}\cdot\left(\operatorname{grad}\varphi + \dfrac{\partial\mathbf{A}}{\partial t} + \operatorname{rot}\mathbf{C}\right) + \\[2mm] \quad + 4\alpha\left(\dfrac{\partial\psi}{\partial t} + \operatorname{div}\mathbf{C}\right) + 4\beta\left(\dfrac{\partial\varphi}{\partial t} + \operatorname{div}\mathbf{A}\right) - 4\,(\alpha\beta - \mathbf{E}\cdot\mathbf{B}) + \\[2mm] \quad + 4\,(\varphi\sigma + \psi\rho - \mathbf{A}\cdot\boldsymbol{m} - \mathbf{C}\cdot\boldsymbol{j})\,. \end{cases}$$

The formulation sketched above can be considered as a generalization of the classic variational formulation [4 ; 5 ; 6], but we remark that both the field equations and the relation between field and potential are obtained by the same functional, whereas the classic formulation generally gives one group of equations and the solutions of the second group, expressed in terms of the potentials. As for the wave equation (1.7) for the four-potential A_ν, it can be formally deduced from the functional:

$$\mathscr{G}(a) \equiv -\tfrac{1}{2}\int_\Omega (A_\mu{}^{/\nu} A^\mu{}_{/\nu} + 2 j_\mu A^\mu)\, d\Omega\,, \tag{2.14}$$

which is functionally the same as the classic functional, A_ν and j_ν being given by the generalized expressions (1.1).

3. The stress-energy tensor

Among various procedures for constructing a stress-energy tensor, it is well-known [7, ch. 4], that one can start from the Lagrangian of the pure field to obtain a tensor:

$$T^{\mu\nu} \equiv -g^{\mu\nu}\mathscr{L} + g^{\lambda\nu} A_{\sigma/\lambda} \frac{\partial \mathscr{L}}{\partial (A_{\sigma/\mu})}\,, \tag{3.1}$$

whose divergence vanishes if the field equations are satisfied; thus by means of the Lagrangian of the functional (2.12) and of the relation (1.6) between field and potential, we have:

$$T^{\mu\nu} \equiv -\tfrac{1}{2} g^{\mu\nu} H^{\alpha\beta} H_{\alpha\beta} + A_\lambda{}^{/\nu} [H^{\mu\lambda} - H^{\lambda\mu} + g^{\lambda\mu} H^{\sigma}_{\cdot\sigma} + i\varepsilon^{\rho\sigma\lambda\mu} H_{\rho\sigma}]\,. \tag{3.2}$$

However, in contrast with the case of classic electromagnetism, it does not seem straightforward to symmetrize the tensor (3.2) and to express it only in terms of the field by means of eq. (1.6); therefore a different procedure can be adopted, making use of the field equations directly, which gives the same formulation as the one previously obtained from the vector equations for the Pauli algebra [1].

To this end, we consider the following condition, which is implied by eqs. (1.4)

$$H_{\mu\nu}{}^{/\mu} H^{*\nu\sigma} + H^{*}_{\mu\nu}{}^{/\mu} H^{\nu\sigma} \equiv j_\nu H^{*\nu\sigma} + j^{*}_\nu H^{\nu\sigma}, \tag{3.3}$$

where $H^{*}_{\mu\nu}$ and j^{*}_μ are the complex conjugate field and current respectively: by taking into account the following identity (independent of the field equations) for $H_{\mu\nu}$,

$$H^{*\mu\nu/\sigma} H_{\sigma\mu} = H^{*\mu\nu} H_{\sigma\mu}{}^{/\sigma}\,, \tag{3.4}$$

and by defining the generalized stress-energy tensor $\theta^{\mu\nu}$ and four-force K^ν

$$\theta^{\mu\nu} \equiv \tfrac{1}{2}(H^{*\mu\sigma} H_\sigma^{\cdot\nu} + H^{\mu\sigma} H^{*\nu}_\sigma) \quad ; \quad K^\nu \equiv \tfrac{1}{2}(H^{*\sigma\nu} j_\sigma + H^{\sigma\nu} j^{*}_\sigma)\,, \tag{3.5}$$

one easily verifies that eq. (3.3) corresponds to four real equations of the following form:

$$\theta^{\mu\nu}{}_{/\mu} = K^{\nu} \,. \tag{3.6}$$

In fact, by (3.4) and (3.5) it is

$$\begin{aligned} \theta^{\mu\nu}{}_{/\mu} &= \operatorname{Re}(H^{*\mu\sigma} H_{\sigma}^{\cdot\nu})_{/\mu} = \operatorname{Re}(H^{*\mu\sigma}{}_{/\mu} H_{\sigma}^{\cdot\nu}) + \operatorname{Re}(H^{*}_{\lambda\sigma} H^{\sigma\nu/\lambda}) = \\ &= \operatorname{Re}(j^{*}_{\mu} H^{\mu\nu}) + \operatorname{Re}(H^{*\sigma\nu} H_{\mu\sigma}{}^{/\mu}) = 2\operatorname{Re}(j^{*}_{\mu} H^{\mu\nu}) = K^{\nu}. \end{aligned} \tag{3.7}$$

An energetic formulation of the field equations can thus be obtained from eqs. (1.4) more easily than from eqs. (2.1): at last one can directly verify that

$$\theta^{\mu\nu} \equiv \left[\begin{array}{c|c} \varepsilon' & \boldsymbol{s}' + \boldsymbol{l} \\ \hline \boldsymbol{s}' & -p'_{ki} \end{array}\right], \tag{3.8}$$

i.e. the non-symmetric form already obtained in [1]: ε', $\boldsymbol{s}'$, p'_{ki} are the generalized field energy, the generalized Poynting vector and the generalized stress tensor, whereas the vector $\boldsymbol{l}$ has no clear physical meaning. Therefore we conclude by recalling that $\theta^{\mu\nu}$ can be made symmetric by the method used in [1], where the tensor made symmetric is indicated by $T^{\prime\prime\mu\nu}$ in a completely analogous way as in the classic case, $T^{\prime\prime\mu\nu}$ can thus be expressed only by means of the energy, the Poynting vector and the stress.

4. FINAL REMARKS

From the analysis made in [1] and in this paper, the conclusion can be drawn that the properties of the classic electromagnetic field can be maintained: the equations can be given in a vector or a space-time tensor formulation, and can be derived from an action principle. Under the hypothesis that the field and the current are represented by the most general elements of the Pauli or Dirac algebra, the following properties are valid:

1) the generalized energy ε' is positive definite;

2) the Poynting vector $\boldsymbol{s}'$ is parallel to the wave vector;

3) the stress tensor is symmetric: its linear invariant is the opposite of the field energy ε';

4) the stress-energy tensor is symmetric, and $T^{\prime\prime 00} > 0$.

The charge conservation and the non-existence of magnetic charges are not implied. However, one has to take into account also the experimental fact that magnetic charges seem to have never been observed and that the motion of the electric charged current density can be described only in terms of a conservation law.

As it can be easily verified, if the above dissimmetry between electric and magnetic currents is required and suitable hypotheses are assumed on the uniqueness of the solution of the generalized equations, it follows that $\alpha = \beta = 0$: the generalized field reduces again to the classic field.

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