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Two theorems characterizing increasing k-set contraction mappings

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Topologia. — *Two theorems characterizing increasing k-set contraction mappings.* Nota di KANHAYA LAL SINGH, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Vengono caratterizzati certi tipi di contrazioni, facendone fra l'altro applicazione per ritrovare alcuni risultati di Yamamuro [9].

The notion of measure of noncompactness was introduced by C. Kuratowskii [1]. It was Darbo [2] who defined the concept of k-set contraction using the notion of measure of noncompactness. The concept of densifying mappings was introduced by Furi and Vignoli and Nussbaum independently in [3] and [4] respectively. The concept of 1-set contraction was given by Petryshyn [5]. Nussbaum using the measure of noncompactness of Kuratowskii [1] developed the degree theory for k-set contraction with $k < 1$, and later extended it to densifying mappings. The degree theory for 1-set contraction was defined by Petryshyn [6]. There are several other kinds of measure of noncompactness developed by Sadovskii [5]. The elegant work of Petryshyn [7] and [8] contains applications of densifying as well as of 1-set contraction mappings.

In the present paper we prove two theorems using the concept of k-set contraction with $k<1$, characterizing the increasing k-set contraction mappings. An application of this concept is given to the degree theory of k-set contraction mappings. The results of Yamamuro [9] are obtained as Corollaries.

DEFINITION 1.1. (C. Kuratowskii [1]). Let X be a real Banach space. Let D be a bounded subset of X. The *measure of noncompactness of* D, denoted by $\gamma(D)$, is defined as follows:

$\gamma(D) = \inf \{\varepsilon > 0/D$ can be covered by a finite number of subsets of diameter $< \varepsilon\}$;

$\gamma(D)$ has the following properties:

(1) $0 \le \gamma(D) \le \delta(D)$, where $\delta(D)$ is the diameter of D,

(2) $\gamma(D) = 0$ if and only if D is precompact (i.e. $\bar{D}$ is compact),

(3) $\gamma(\bar{D}) = 0$ if and if $\gamma(D) = 0$,

(4) $\gamma(C \cup D) = \max\{\gamma(C), \gamma(D)\}$,

(5) $C \subset D$ implies $\gamma(C) \le \gamma(D)$,

(6) $\gamma(C + D) \le \gamma(C) + \gamma(D)$, where $C + D = \{c + d/c$ in C and d in D$\}$,

(7) $\gamma(S(D, r)) < \gamma(D) + 2r$, where $S(D, r) = \{x$ in $X/d(x, D) < r\}$.

(*) Nella seduta del 15 novembre 1975.

Closely related to the notion of measure of noncompactness is the concept of k-set contraction introduced by Darbo [2] as follows:

DEFINITION 1.2. Let X be a real Banach space. Let $T : X \to X$ be a continuous mapping. Then T is said to be a k-set *contraction* if for any bounded but not precompact subset D of X we have

$$\gamma(T(D)) \leq k\gamma(D)$$

for some $k \geq 0$. In case $\gamma(T(D)) < \gamma(D)$, $(\gamma(T(D)) \leq \gamma(D))$ for any bounded but not precompact subset D of X such that $\gamma(D) > 0$, the T is called *densifying* (*1–set contraction*).

REMARK 1.1. In our discussion we will restrict k in Definition 1.2 to satisfy $0 \leq k < 1$.

DEFINITION 1.3. A mapping $T : D \to X$ is said to be a k-set contraction vector field if T can be represented as $T(x) = I(x) - F(x)$, where $I(x)$ is the identity mapping and $F : D \to X$ is a k-set contraction.

DEFINITION 1.4. A mapping $T : D \to X$, where D is some bounded subset of X, is said to be *Fréchet-differentiable* at x in D if there exists a continuous linear mapping $T_{x_0} : X \to X$ such that

$$T(x + x_0) - T(x) = T_{x_0} + W(x, x_0) \qquad \text{for all } x \text{ in X, where}$$

$$\lim_{\|x\| \to 0} \frac{\|W(x, x)\|}{\|x\|} \to 0. \tag{1}$$

The linear mapping T_{x_0} is called the *Fréchet-derivative* of T and is denoted by $T'(x_0)$.

DEFINITION 1.5. Let X be a real Banach space. Let D be an open bounded subset of X. Let $\bar{D}$ be the closure of D. A mapping $T : \bar{D} \to X$ is said to be (ε, δ)-uniformly increasing at x_0 in D if there exist numbers $\varepsilon > 0$ and $\delta > 0$ such that the following conditions are satisfied:

(1) $\|x\| < \delta$ implies $x + x_0$ in D,

(2) $\|T(x + x_0) - T(x_0) - \alpha x\| \geq \varepsilon \|x\|$ if $\alpha \leqq 0$ and $0 < \|x\| < \delta$.

LEMMA 1.1. *The Fréchet-derivative of a k-set contraction with $k < 1$ is a k-set contraction with $k < 1$.*

The proof of Lemma 1.1 may be found either in Nussbaum [10], pp. 191, or in Sadovskii [5], pp. 103.

LEMMA 1.2. *Let X be a real Banach space. Let D be an open, bounded subset of X. Let $T : \bar{D} \to X$ be a Fréchet-differentiable k-set contraction vector field with $k < 1$. Let the range of $T'(x_0)$, the Frécthet-derivative of T at x_0, be compact. Then every eigenvalue of $T'(x_0)$ is positive if and only if T is (ε, δ)-uniformly increasing at x_0 for some $\varepsilon > 0$ and $\delta > 0$.*

Proof. Suppose that every eigenvalue of $T'(x_0)$ is positive, but T is not (ε, δ)-uniformly increasing at x_0 for $\varepsilon > 0$ and $\delta > 0$. Then we can find a sequence of elements x_n $(n = 1, 2, 3, \cdots)$ and numbers α_n such that

$$(1) \qquad \| T(x_0 + x_n) - T(x_0) - \alpha_n x_n \| < (1/n) \| x_n \|, \quad \alpha_n < 0 \quad \text{and} \quad 0 < \| x_n \| < (1/n).$$

Let us denote for simplicity $T(x + x_0) - T(x_0)$ by $T_{x_0}(x)$. Thus we can write (1) as

$$\| T_{x_0}(x_n) - \alpha_n x_n \| < (1/n) \| x_n \|, \quad \alpha_n < 0 \quad \text{and} \quad 0 < \| x_n \| < (1/n).$$

Now we wish to show that the sequence $\{\alpha_n\}$ is bounded. Indeed, taking into account (1) and the linearity of $T'(x_0)$, we have

$$\begin{aligned} \alpha_n &= \| \alpha_n x_n \| / \| x_n \| \leq 1 \{ \| T_{x_0}(x_n) - \alpha_n x_n \| + \| T_{x_0}(x_n) \| \} / \| x_n \| \\ &\leq (1/n) + \| T_{x_0}(x_n) \| / \| x_n \| \leq (1/n) + T'(x_0)(x_n / \| x_n \|) + \| W(x_0, x_n) / \| x_n \| \\ &\leq (1/n) + \| T'(x_0) \| + \| W(x_0, x_n) \| / \| x_n \|, \end{aligned}$$

where the right-hand side is bounded because of (1) of Definition 1.4. ($\| T'(x_0) \|$ is the norm of the linear mapping $T'(x_0)$, since $T'(x_0)$ is continuous and the linear mapping $\| T'(x_0) \|$ is finite); therefore there exists a subsequence $\{\alpha_m\}$ of $\{\alpha_n\}$ such that $\lim_{m \to \infty} \alpha_m = \alpha_0$ for some non-positive number α_0.

Let us first observe that $x - T(x)$ is also a k_1-set contraction with $k_1 < 1$. Indeed let $F(x) = x - T(x)$, then F being the difference of two continuous mappings is certainly continuous. Let A be any bounded but not precompact subset of D. Then

$$\gamma(F(A)) \leq \gamma(A) - k\gamma(A) = (1 - k)\gamma(A) = k_1 \gamma(A), \quad \text{where} \quad 0 < k_1 < 1.$$

Moreover from Lemma 1.1. we conclude that the *Fréchet-derivative* F, $F'(x_0) = I - T'(x_0)$ is also a k_1-set contraction with $0 < k_1 < 1$.

Let $x'_m = (x_m / \| x_m \|)$. Then $\| x'_m \| = 1$, that is x'_m is bounded. Since by hypothesis the range of $F'(x_0)$ is compact, we can infer the existence of a subsequence $\{x'_k\}$ of $\{x'_m\}$ such that $\lim_{k \to \infty} F'(x_0)(x'_k) = x_1$ for some element x_1.

On the other hand by (1) and by (1) of Definition 1.4 we have

$$\begin{aligned} (4) \qquad & \lim_{k \to \infty} \{ (1 - \alpha_k) x'_k - F'(x_0) x'_k \} = \lim_{k \to \infty} \{ x'_k - F'(x_0) x'_k - \alpha_k x'_k \} \\ &= \lim_{k \to \infty} \{ T'(x_0) x'_k - \alpha_k x'_k \} = \lim_{k \to \infty} (1/\| x_k \|) \{ T'(x_0) x'_k - \alpha_k x'_k \} \\ &= \lim_{k \to \infty} (1/\| x_k \|) \{ T'(x_0) x_k - \alpha_k x_k \} = \lim_{k \to \infty} (1/\| x_k \|) \{ T'(x_0) x_k - T_{x_0}(x_k) \} + \\ &+ \lim_{k \to \infty} (1/\| x_k \| \{ T_{x_0}(x_k) - \alpha_k x_k \} = 0. \end{aligned}$$

Therefore it follows from (3) that

$$\lim_{k\to\infty} (1 - \alpha_k)\, x_k = x_1 .$$

Hence from (2) we have

$$\lim_{k\to\infty} x_k' = (1/(1 - \alpha_0))\, x_1$$

which implies that $\| x_1 \| = 1 - \alpha_0$ and by (4) we have

$$T'(x_0)\,(x_1/(1 - \alpha_0)) = \lim_{k\to\infty} T'(x_0)\, x_k = \lim_{k\to\infty} \alpha_k\, x_k' = \alpha_0\, x_1/(1 - \alpha_0).$$

This means that α_0 is a nonpositive eigenvalue of $T'(x_0)$.

Conversely let us assume that T is (ε , δ)-uniformly increasing at x_0 for some $\varepsilon > 0$ and $\delta > 0$. Suppose α is an eigenvalue of $T'(x_0)(x)$, i.e.

$$T'(x_0)\,(x) = \alpha x.$$

Since $T'(x_0)$ is linear, by (4) we have

$$T'(x_0)\,(\eta x) = \alpha\eta\,(x) \qquad \text{for every number } \eta.$$

Hence by (1) of Definition 1.5 we conclude that there exists $\delta_1 > 0$ such that $\delta_1 < \delta$ and $\| W(x_0 , \eta x) \| < \varepsilon \mid \eta \mid$ if $\mid \eta \mid < \delta_1$.

Thus for $\mid \eta \mid < \delta_1$ we have

$$\begin{aligned} \| T_{x_0}(\eta x) - \alpha\eta\,(x) \| &= \| T'(x_0)\,(\eta x) + W(x_0 , \eta x) - \alpha\eta x \| \\ &= \| W(x_0 , \eta x) \| < \varepsilon \mid \eta \mid = \varepsilon \| \eta x \| \end{aligned}$$

which by (2) of Definition 1.5 implies that $\alpha > 0$.

LEMMA 1.3. *Let* X *be a real Banach space. Let* $(-T) : \bar{D} \to X$ *be a k-set contraction vector field. Then every eigenvalue of the linear mapping* $T'(x_0)$ *is negative if and only if the mapping* $(-T)$ *is* (ε , δ)*-uniformly increasing at* x_0 *for some* $\varepsilon > 0$, $\delta > 0$ *and the range of* $(-T)'(x_0)$ *is compact.*

Proof. The proof of Lemma 1.3 follows immediately from the proof of Lemma 1.2, since a number is a positive eigenvalue of $T'(x_0)$ if, and only if, it is the absolute value of a negative eigenvalue of $-T'(x_0) = (-T)'(x_0)$.

THEOREM 1.1. *Let* X *be a real Hilbert space. Let* D *be an open bounded subset of* X. *Let* $\pm T : \bar{D} \to X$ *be k-set contraction vector fields on* D. *Let* T *be Fréchet-differentiable at* x_0 *in* D. *Let the Fréchet-derivative* $T'(x_0)$ *of* T *satisfy the following conditions*:

(A) $(T'(x_0)\, x , x) \neq 0$ *if* $x \neq 0$,

(B) *the range of* $T'(x_0)$ *is compact.*

Then either T *or* $-T$ *is* (ε, δ)*-uniformly increasing at* x *in* D *for some* $\varepsilon > 0$ *and* $\delta > 0$.

Proof. Suppose neither T nor —T is (ε, δ)-uniformly increasing at x_0 for any $\varepsilon > 0$ and $\delta > 0$. Then from Lemma 1.2. and Lemma 1.3. we conclude the existence of numbers α_i $(i = 1, 2)$ and elements x_i $(i = 1, 2)$ such that

$$T'(x_0)(x_i) = \alpha_i x_i \ (i = 1, 2)\ (\alpha_1 \geq 0, \alpha_2 \leq 0) \quad \text{and} \quad x_i = 1 \ (i = 1, 2).$$

But by condition (A) α_i are non-zero, therefore $\alpha_1 > 0$ and $\alpha_2 < 0$.

Since x_1 and x_2 are linearly independent we have

$$z(t) = (1 - t)x_1 + tx_2 \qquad (0 \leq t \leq 1).$$

Now

(5) $$z(0) = x_1 \quad \text{and} \quad z(1) = x_2 .$$

Finally we consider the continuous function defined by

$$\varphi(t) = (T'(x_0) z(t)), z(t)) \qquad (0 \leq t \leq 1).$$

Then $\varphi(0) = (T'(x_0)(z(0)), z(0)) = (T'(x_0)(x_1), x_1) = (a_1 x_1, x_1) = \alpha_1 \| x_1 \|^2 > 0$ and $\varphi(1) = (T'(x_0)(z(1)), z(1)) = (T'(x_0)(x_2), x_2) = (\alpha_2 x_2, x_2) = \alpha_2 \| x_2 \|^2 < 0$. Thus there exists t_0 in $(0, 1)$ such that $\varphi(t_0) = 0$, i.e. $(T(x_0)(z(t_0), z(t_0)) = 0$. Since $z(t) \neq 0$, a contradiction to (A). Hence the Theorem.

REMARK 1.2. Let X be a metric space. Let $T : X \to X$ be a continuous mapping. T is said to be a *k-contraction* if $d(T(x), T(y)) \leq kd(x, y)$ for all x, y in X. It follows from Proposition 7, p. 15 in (3) that every k-contraction is a k-set contraction.

EXAMPLE 1.1. An example of a mapping T such that both +T and —T are L-set contractions with $L < 1$ can be obtained by taking $X = R$, the reals, and defining $T : R \to R$ by $T(x) = (3/4)(x)$. Then clearly T satisfies the Lipschitz condition with Lipschitz constant $L = (3/4)$. Hence by Remark 1.2. T is L-set contraction with $L < 1$. Moreover

$$|-T(x) - (-T(y))| = |T(y) - T(x)| = |T(x) - T(y)| = (3/4)|x - y| .$$

Thus —T also satisfies the Lipschitz condition with Lipschitz constant $L = (3/4)$. Therefore —T is also a L-set contraction with $L < 1$. Furthermore $T'(x) = 3/4$ and $(-T)'(x) = -3/4$. Clearily T' and $-T'$ both are 0–set contraction, since both of them satisfy the Lipschitz condition with Lipschitz constant $L = 0$. Obviously the range of T, being finite dimensional, is compact.

DEFINITION 1.6. Let X and Y be two Banach spaces. Let D be a bounded subset of X. A continuous mapping $T : DX \to Y$ is said to be *compact* if $T(D)$ is relatively compact.

REMARK 1.3. A compact mapping is a 0-set contraction. Indeed, let D be any bounded subset of X; then, since T is compact, T (D) is pre-compact. Hence by property (2) of γ we have $\gamma(T(D)) = 0$. Thus T is a 0-set contraction.

COROLLARY 1.1. (Yamamuro). *Let* X *be a real Hilbert space. Let* D *be an open bounded subset of* X. *Let* $\pm T : D \to X$ *be completely continuous vector fields on* D. *If* T *is Fréchet-differentiable at* a *in* D *and the Fréchet-derivative* T' *satisfies the following condition*:

$$(1) \qquad (T'(a)(x), x) \neq 0 \quad if \quad x \neq 0$$

then T *or* —T *is* (ε, δ)-*uniformly increasing at* a *in* D *for some* $\varepsilon > 0$ *and* $\delta > 0$.

DEFINITION 2.1. A mapping $T : \bar{D} \to X$ is said to be *strongly increasing* (or *strongly nondecreasing*) at x_0 in D if there exists a sequence $\{x_n\}$ such that $x_n \neq 0$, $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} \| T_{x_0}(x_n) - \alpha x_n \| / \| x_n \| = 0$ implies $\alpha > 0$ (or $\alpha \geq 0$).

THEOREM 2.1. *Let* X *be a real Banach space. Let* D *be an open, bounded subset of* X. *Let* $T : \bar{D} \to X$ *be a* k-*set contraction vector field with* $k < 1$. *Let us assume that* T *is Fréchet-differentiable at* x_0 *in* D. *Furthermore suppose that the range of* $T'(x_0)$, *the Fréchet-derivative of* T *at* x_0, *is compact. Then* T *is strongly increasing* (*or strongly nondecreasing*) *at* x_0 *if and only if every eigenvalue of the Fréchet-derivative* $T'_{x_0}(0, x)$ *is positive* (*or non-negative*).

Proof. Let us assume that T is strongly increasing at x_0 and λ_1 is an eigenvalue of $T'_{x_0}(0, x)$. Therefore there exists x_1 in X such that

$$T'_{x_0}(0, x) = \lambda_1 x_1 .$$

Since the *Fréchet-derivative* $T'_{x_0}(0, x)$ is linear with respect to x, we have:

$$T'_{x_0}(0, tx_1) = \lambda_1 tx_1 \quad \text{for any number } t.$$

Moreover by definition of *Fréchet-derivative* we have

$$\lim_{t\to\infty} \| T_{x_0}(tx_1) - T'_{x_0}(0, tx_1) \| = 0 \quad \text{or} \quad \lim_{t\to\infty} \| T_{x_0}(tx_1) - \lambda_1 tx_1 \| = 0.$$

Let us first observe that the mapping T is strongly increasing (or strongly nondecreasing) at x_0 in D if and only if the mapping $T_{x_0}(x) = T(x + x_0) - T(x_0)$ is strongly increasing (or strongly non-decreasing) at zero (the zero element of X). Since T_{x_0} is strongly increasing at zero, it follows that $\lambda_1 > 0$.

Conversely let us assume that every eigenvalue of $T'_{x_0}(0, x)$ is positive and there exists a sequence $\{x_n\}$ such that

$$(1) \qquad x_n \neq 0, \lim_{n\to\infty} x_n = 0 \quad \text{and} \quad \lim_{n\to\infty} \| T_{x_0}(x_n) - \alpha x_n \| / \| x_n \| = 0 .$$

Without loss of generality we may assume that $\alpha \neq 1$. Since by assumption T_{x_0} is k-set contraction vector field, we can write T_{x_0} as $T_{x_0}(x) = x - F_{x_0}(x)$, where $F_{x_0}(x)$ is a k-set contraction mapping. Hence from (1) we have

$$(2) \qquad \lim_{n\to\infty} \| (1 - \alpha) x_n - F_{x_0}(x_n) \| / \| x_n \| = 0 .$$

From (2) we conclude that

$$\lim_{n\to\infty} \| (1 - \alpha) x_n - F'_{x_0}(x_n) \| / \| x_n \| = 0$$

where $F'_{x_0}(0, x)$ is the *Fréchet-derivative* of $F_{x_0}(x)$ at zero. The existence of such a derivative is guaranteed by the existence of *Fréchet-derivative* of $T_{x_0}(x)$ at zero. Let $(y_n = x_n / \| x_n \|)$. Using the linearity of $F'_{x_0}(0, x)$ with respect to x we have

$$(3) \qquad \lim_{n\to\infty} \| (1 - \alpha) y_n - F'_{x_0}(0, y_n) \| = 0 .$$

Since $\| y_n \| = \| x_n \| / \| x_n \| = 1$, $\{y_n\}$ is bounded. Now using the linearity of $F'_{x_0}(0, x)$ and the assumption that the range of the *Fréchet-derivative* is compact, we conclude the existence of a subsequence $\{y_{n_i}\}$ such that

$$\lim_{i\to\infty} F_{x_0}(0, y_{n_i}) = y_0$$

for some y_0 in X. Using the continuity of the norm from (3) we have

$$\lim_{i\to\infty} (1 - \alpha) y_{n_i} = y_0 .$$

Therefore $\| y_0 \| = \| 1 - \alpha \| \neq 0$ and

$$y_0 - T_{x_0}(0, y_0) = F'_{x_0}(0, y_0) = (1 - \alpha) y_0$$

so that α is an eigenvalue of $T'_{x_0}(0, x)$, hence we conclude that $\alpha > 0$.

DEFINITION 2.2. Let X be a real Banach space. Let D be an open, bounded subset of X. A mapping $T : D \to X$ is said to be (δ)-*increasing* at x_0 in D if there exists a number $\delta > 0$ such that

(1) $x + x_0$ in D if $\| x \| < \delta$;

(2) $T(x + x_0) - T(x_0) \neq \alpha x$ if $\alpha \leq 0$ and $0 < \| x \| < \delta$.

In [30] we proved the following Theorem:

THEOREM K. *Let* X *be a real Banach space. Let* D *be an open, bounded subset of* X. *Let* $T : \bar{D} \to X$ *be a k-set contraction vector field with* $k < 1$. *If* T *is* (δ)-*increasing at* x_0 *in* D, *then for any* $\delta_1 > 0$ *such that* $\delta_1 < \delta$ *we have* *Deg* $(0, B(0, \delta_1), T_{x_0}) = 1$, *where* $B(0, \delta_1)$ *denotes the ball of radius* δ_1 *with center at zero.*

Since every (ε, δ)-uniformly increasing mapping at x_0 in D is also (δ)-increasing at x_0 in D, as a corollary Theorem 1.1 we have

COROLLARY 2.1. *Let* X *be a real Hilbert space. Let* D *be an open, bounded subset of* X. *Let* $\pm T : \bar{D} \to X$ *be k-set contraction with* $k < 1$. *Suppose that*

T *is Fréchet-differentiable at* x_0 *in* D *and* $T'(x_0)$ *satisfies conditions* (A) *and* (B) *of Theorem* 1.1, *then there exists* $\delta_1 > 0$ *such that*

$$\text{Deg}\,(0\,,\,B\,(0\,,\,\delta_1\,,\,T_{x_0}) = 1 \qquad \text{for any } \delta_1 > 0 \text{ such that } 0 < \delta_1 < \delta$$

or $$\text{Deg}\,(0\,,\,B\,(0\,,\,\delta_1)\,,\,-T_{x_0}) = 1 \quad \text{for any } \delta_1 > 0 \text{ such that } 0 < \delta_1 < \delta.$$

As a final result we deduce the following theorem of Yamamuro as a Corollary of Theorem 2.1.

COROLLARY 2.2. *Let* E *be a real Banach space. Let* G *be an open, bounded subset of* E. *Let* $f : \overline{G} \to E$ *be a completely continuous vector field so that the set* $F(\overline{G})$ *is contained in a compact set, where* $F(x) = x - f(x)$. *Let us assume that* f *is Fréchet-differentiable at* a *in* G. *Then* f *is strongly increasing (or strongly non-decreasing) at* a *if, and only if, every proper value of the Fréchet-derivative is positive (or non-negative).*

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