Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Aiden A. Bruen, Joseph A. Thas

Flocks, chains and configurations in finite geometries

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **59** (1975), n.6, p. 744–748.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1975_8_59_6_744_0>

Geometrie finite. — *Flocks, chains and configurations in finite geometries.* Nota di AIDEN A. BRUEN e JOSEPH A. THAS, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Lo studio dei sistemi di cerchi [o sezioni piane contenenti più di un punto] di un ovaloide [O (q^2+1)-calotta] di un $S_{3,q}$ ha utili applicazioni nella teoria dei piani di traslazione. Qui sistemi siffatti vengono investigati con particolare riguardo al caso in cui i piani dei loro cerchi escono da un punto non situato sull'ovaloide, assieme alla configurazione formata dai poli di tali piani rispetto all'ovaloide.

1. INTRODUCTION

An ovoid O of the threedimensional projective space PG $(3, q)$, $q > 2$, is a set of $q^2 + 1$ points no three of which are collinear. The circles of O are the sets $P \cap O$, where P is a plane of PG $(3, q)$, with $|O \cap P| > 1$. The circles of a non-singular ruled quadric Q of PG $(3, q)$ are, by definition, the irreducible conics on Q. In what follows O will always denote an ovoid and Q a non-singular ruled quadric.

The study of sets of circles on O and Q is important for the theory of translation planes (see also section 2 below). If the planes of the circles of such a set all meet in one point $p \notin O$ or Q, then their poles (with respect to O or Q) all lie in the polar plane P of p. Moreover these poles constitute an interesting configuration of points with respect to the circle $P \cap O$ or $P \cap Q$. This note is mainly concerned with such configurations of points.

2. FLOCKS

A *flock* of O (resp. Q) is a set F of $q - 1$ (resp. $q + 1$) mutually disjoint circles. If L is a line of PG $(3, q)$ which has no point in common with O (resp. Q), then the circles $P \cap O$ (resp. $P \cap Q$), where P is a plane containing L with $|P \cap O| > 1$ (resp. where P is a plane containing L), constitute a so-called linear flock of O (resp. Q).

That each flock of the ovoid O is linear was proved by J. A. Thas for q even [3] and by W. F. Orr in the odd case [2]. Thas [4] also proved that each flock of the non-singular ruled quadric Q of PG $(3, q)$, q even, is linear, and that for each odd q the quadric Q has a non-linear flock. (We should also mention here that using the hyperquadric of Klein, it is possible to prove that with each non-linear flock of Q there corresponds a non-desarguesian transla-

(*) Nella seduta del 13 dicembre 1975.

tion plane of order q^2). As an application of theorems about flocks we state an interesting result concerning configurations of points in the plane PG $(2, q)$.

THEOREM. (*a*) *Let* C *be an oval of* PG $(2, q)$, $q > 2$, *which can be embedded in an ovoid of* PG $(3, q)$ (*e.g. an irreducible conic*). *If* $F = \{x_1, x_2, \cdots, x_{q-1}\}$ *is a set of* $q - 1$ *points of* PG $(2, q)$ — C, *such that any line* $x_i x_j$, $i \neq j$, *is a secant of* C, *then the points of* F *all lie on one secant of* C.

(b) *Let* C *be an irreducible conic of* PG $(2, q)$, q *even. If* $F = \{x_1, x_2, \cdots, x_{q+1}\}$ *is a set of* $q + 1$ *points such that any line* $x_i x_j$, $i \neq j$, *is an exterior line of* C, *then* F *is an exterior line of* C (= *non-secant of* C).

Proof. (*a*) Let C be embedded in an ovoid O of PG $(3, q)$. The polar planes $P_1, P_2, \cdots, P_{q-1}$ of $x_1, x_2, \cdots, x_{q-1}$ with respect to O, intersect O in $q - 1$ mutually disjoint circles. These circles constitute a flock F^* of O. As F^* is linear the planes P_i all pass through one exterior line L of O. Consequently their poles x_i all lie on one secant of O and hence on one secant of C.

(b) Let C be embedded in a non-singular ruled quadric Q of PG $(3, q)$ (q even). The polar planes $P_1, P_2, \cdots, P_{q+1}$ of $x_1, x_2, \cdots, x_{q+1}$ with respect to Q, intersect Q in $q + 1$ mutually disjoint circles. These circles constitute a flock F^* of Q. As q is even, F^* is linear, and so the planes P_i all pass through one exterior line L of Q. Consequently their poles x_i all lie on one exterior line of Q. We conclude that F is an exterior line of C.

COROLLARY. *Let* O *be an elliptic quadric of* PG $(3, q)$, q *even. If* F *is a set of* $q + 1$ *circles, any two of which have two points in common, and if furthermore the planes of these circles all meet in one point* $p \notin O$, *then* F *is a pencil of circles* (*i.e. the* $q + 1$ *circles all meet in two fixed points*).

Proof. The poles (with respect to O) of the $q+1$ planes containing the elements of F, all lie in the polar plane P of p. The line joining any two of these poles is an exterior line of the irreducible conic $P \cap O$. Since q is even, the set F^* of these poles is an exterior line of PO by our previous theorem. Consequently the planes of the $q+1$ circles of F all contain one fixed secant of O. We conclude that F is a pencil of circles of the quadric O.

3. CHAINS OF CIRCLES AND THE CORRESPONDING CONFIGURATIONS IN THE PLANE

In [1] A.A. Bruen studies maximal families F of circles on an elliptic quadric O of PG $(3, q)$, q odd, having the following two properties:

(*a*) Any two circles of F have two distinct points in common;

(*b*) No three circles of F have a point in common.

It is easy to see that F contains at most $(q+3)/2$ circles, i.e. $|F| \leq (q+3)/2$. If $|F| = (q+3)/2$ each point on a circle of the set F is contained in exactly two circles of F. Such a set of $q + 3/2$ circles having properties (*a*) and (*b*) above is called a *chain of circles*. In [1] Bruen constructs a chain in the cases $q = 3, 5, 7$ and shows that with certain chains of circles there correspond new translation planes of order q^2. For further details we refer to [1].

Let O be an elliptic quadric of PG $(3, q)$, q odd, and let $F = \{C_1, C_2, \cdots, C_{(q+3)/2}\}$ be a chain of O. Suppose further that the planes P_i of C_i, $i = 1, 2, \cdots, q + 3/2$, all meet in one point p $(p \notin O)$. Then the poles $x_1, x_2, \cdots, x_{(q+3)/2}$ of the planes $P_1, P_2, \cdots, P_{(q+3)/2}$ all lie in the polar plane P of p. Moreover the set $F^* = \{x_1, x_2, \cdots, x_{(q+3)/2}\}$ has the following properties:

(i) any line $x_i x_j$, $i \neq j$, is an exterior line of the irreducible conic $C = P \cap O$;

(ii) F^* is an $(q+3)/2$-arc of the plane P (i.e. no three points of F^* are collinear).

Conversely, we consider in PG $(2, q)$, q odd, a set F^* of $(q+3)/2$ points for which (i) and (ii) are satisfied, where C is an arbitrary irreducible conic. We embed C in an elliptic quadric O of PG $(3, q)$, and we consider the polar planes of the elements of F^*. These $(q+3)/2$ planes intersect O in $(q+3)/2$ circles, which constitute a chain of O (moreover the planes of the $(q+3)/2$ circles of the chain all meet in one point). Consequently it is of interest to construct in PG $(2, q)$, q odd, sets F^* for which (i) and (ii) are satisfied. Here we shall only consider the cases $q = 3$ and 5. The general case is being investigated by the authors and will be treated elsewhere.

THEOREM. *Let* C *be an irreducible conic of the projective plane* PG $(2, q)$, $q = 3$ *or* 5. *Then there exists a* $q + 3/2$-*arc* $F^* = \{x_1, x_2, \cdots, x_{(q+3)/2}\}$ *in* PG $(2, q)$, *such that any line* $x_i x_j$, $i \neq j$, *is an exterior line of* C. *Moreover any two such sets* F^* *are equivalent with respect to the group of collineations whih leave* C *invariant.*

Proof. Let $q = 3$. Then $C = \{y_1, y_2, y_3, y_4)$ is a set of four points, no three of which are collinear. If x_1, x_2, x_3 are the diagonal points of the complete quadrangle C, then it is easy to check that $\{x_1, x_2, x_3\}$ is the unique set F^* with the desired properties.

Now we suppose that $q = 5$. Suppose that $F^* = \{x_1, x_2, x_3, x_4\}$ is a 4-arc of PG $(2,5)$, such that any line $x_i x_j$, $i \neq j$, is an exterior line of the conic $C = \{y_1, y_2, \cdots, y_6\}$. We shall prove that the diagonal points z_1, z_2, z_3 of the quadrangle F^* are exterior points of C, and that z_1, z_2, z_3 is a self-polar triangle with respect to C.

An exterior point of C is on two exterior lines of C, and an interior point of C is on three exterior lines of C. Since $x_1 x_2, x_1 x_3, x_1 x_4$ are exterior lines, the point x_1 is an interior point. In fact, each of x_1, x_2, x_3, x_4 are interior points. Consequently there are exactly 12 secants of C which have a point in common with F^*. Since C has exactly 15 secants there are at most three secants of C which have a point in common with $\{z_1, z_2, z_3\}$. Since z_i, $i = 1, 2, 3$, is on at least two secants, it follows immediately that $z_1 z_2, z_2 z_3, z_3 z_1$ are secants and that these are the only secants having a point in common with $\{z_1, z_2, z_3\}$. Hence z_1, z_2, z_3 are exterior points. Let $z_i z_j \cap C = \{u_k, u_k'\}$, where $\{i, j, k\} = \{1, 2, 3\}$. Evidently $C = \{u_1, u_1', u_2, u_2', u_3, u_3'\}$. Since $z_i z_j$ and $z_i z_k$, $\{i, j, k\} = \{1, 2, 3\}$, are the secants through z_i, the lines $z_i u_i, z_i u_i'$ are the tangents through z_i. So $u_i u_i' = z_j z_k$ is the polar line of z_i. We conclude that $z_1 z_2 z_3$ is a self-polar triangle with respect to C. Now we shall prove that F^* is uniquely defined by each of its diagonal points.

Consider the diagonal point z_i of F^*. It is an exterior point. Then z_j, z_k are the exterior points of the polar line of z_i with respect to C. The points x_1, x_2, x_3, x_4 are the intersections of the two exterior lines on z_i with the two exterior lines on z_j. So F^* is uniquely defined by z_i. Since the group G of collineations which leave C invariant is transitive on the set of exterior points, we conclude that any two such sets F^* are equivalent with respect to G.

Finally we prove that F^* exists. Let $GF(5) = \{0, 1, 2, 3, 4\}$ and let C be the irreducible conic with equation $x^2 + y^2 + z^2 = 0$. Consider the exterior point $(0, 0, 1)$. The exterior points on the polar line $z = 0$ of $(0, 0, 1)$ are the points $(0, 1, 0)$ and $(1, 0, 0)$. The exterior lines containing $(0, 0, 1)$ (resp. $(1, 0, 0)$, resp. $(0, 1, 0)$) are $y = x$ and $y = -x$ (resp. $z = y$ and $z = -y$, resp. $x = z$ and $x = -z$). These six exterior lines are exactly the six sides of the complete quadrangle with vertices $(1, 1, 1)$, $(1, 1, -1)$, $(1, -1, -1)$, $(1, -1, 1)$. Consequently $F^* = \{(1, 1, 1), (1, 1, -1), (1, -1, -1), (1, -1, 1)\}$ has the desired properties.

COROLLARY 1. *Each elliptic quadric* O *of* PG $(3, q)$, $q = 3$ *or* 5, *possesses a chain with the property that the planes of the* $(q + 3)/2$ *circles of the chain all meet in one point.*

COROLLARY 2. *Each non-singular ruled quadric* Q *of* PG $(3, q)$ $q = 3$ *or* 5, *possesses a set of* $(q + 3)/2$ *mutually disjoint circles (i.e. a partial flock of size* $(q + 3)/2$*) no three of which are contained in a linear flock and such that the planes of these circles all meet in one point.*

Proof. Let C be a circle of Q and let $F^* = \{x_1, x_2, \cdots, x_{(q+3)/2}\}$ be a set of points which has the properties (i) and (ii) with respect to the conic C, in the plane corresponding to C. If $P_1, P_2, \cdots, P_{(q+3)/2}$ are the polar planes of $x_1, x_2, \cdots, x_{(q+3)/2}$ with respect to Q, then (i) the planes P_i all meet in the pole of the plane containing C (ii) the circles $P_i \cap Q$ constitute a partial flock of order $(q + 3)/2$ (iii) no three circles $P_i \cap Q$ are contained in a linear flock (this follows from the fact that F^* is a $(q + 3)/2$-arc).

REFERENCES

[1] A. A. BRUEN – *Inversive geometry and some new translation planes* (submitted).

[2] W. F. ORR (1973) – *The Miquelian inversive plane* IP (q) *and the associated projective planes*, Thesis submitted to obtain the degree of Doctor of Philosophy at the University of Wisconsin.

[3] J. A. THAS (1972) – *Flocks of finite egglike inversive planes*, «C.I.M.E.», II ciclo, Bressanone», 189–191.

[4] J. A. THAS – *Flocks of non-singular ruled quadrics in* PG $(3, q)$, «Accad. Naz. Lincei» (to appear).