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A note on 2—dimenstonal parafactorial rings

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Geometria algebrica. — *A note on 2-dimensional parafactorial rings* (*). Nota di FERDINANDO MORA e LORENZO ROBBIANO (**), presentata (***) dal Socio E. TOGLIATTI.

RIASSUNTO. — Siano dati una superficie algebrica X definita su un corpo algebricamente chiuso e un punto chiuso x di X: in questo lavoro essenzialmente si prova che la proprietà « ogni fascio invertibile su $X - \{x\}$ si estende in modo unico ad un fascio invertibile su X » implica la fattorialità dell'anello locale in x.

The purpose of this work is essentially the following.

Suppose we are given an algebraic surface which is defined over an algebraically closed field and pick a point on it; if every invertible sheaf on the complement of the point can be uniquely extended to an invertible sheaf on the whole surface, then the local ring at the point is factorial.

For the proof we use a criterion of Grothendieck which characterizes the parafactoriality of 2-dimensional local rings (Lemma 1). On the other hand our result can be used to disprove a remark of [2] asserting that a certain ring is parafactorial (Remark 2).

In this paper all rings are supposed to be commutative, noetherian and with identity.

If $(A, \mathfrak{m}, k)$ is a local ring, we denote by U_A the punctured spectrum of A, i.e. the scheme $(\operatorname{Spec}(A) - \{\mathfrak{m}\}, \mathcal{O})$, where $\mathcal{O}$ is the sheaf induced on $\operatorname{Spec}(A) - \{\mathfrak{m}\}$ by the canonical one of Spec (A). We recall the following

DEFINITION. A local ring $(A, \mathfrak{m}, k)$ is said to be parafactorial if depth $(A) \geq 2$ and $\operatorname{Pic}(U_A) = 0$.

LEMMA 1 ([2] 21.13.9.vi). *Let* $(A, \mathfrak{m}, k)$ *be a local ring such that* $\dim(A) = 2$. *Then* A *is parafactorial and not factorial iff the following conditions are satisfied*:

a) A *is Cohen-Macaulay*;

b) A *is a domain and if* A' *denotes its integral closure*, A' *is factorial and a finite* A-*algebra*;

c) *If* $\mathfrak{F}$ *denotes the conductor of* A *in* A', $B = A/\mathfrak{F}$, $B' = A'/\mathfrak{F}$, *then* $\dim(B) = 1$ *and the canonical morphism* $\operatorname{Div}(B) \to \operatorname{Div}(B')$ *is surjective.*

Furthermore the previous conditions imply that B' (*hence* B) *is reduced and the canonical map* $\operatorname{Spec}(B') \to \operatorname{Spec}(B)$ *is bijective.*

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LEMMA 2. *Let* $R \subseteq S$ *be Dedekind domains such that*

a) S *is a finite* R-*module*;

b) *For every* $\mathfrak{m} \in \mathrm{Max}(R)$ *and for every* $\mathfrak{M} \in \mathrm{Max}(S)$, $R/\mathfrak{m} \simeq S/\mathfrak{M}$;

c) If $K(R)$, $K(S)$ *denote the fields of fractions* of $R, S, n = [K(S) : K(R)] > 1$.

Then the canonical morphism $i : \mathrm{Div}(R) \to \mathrm{Div}(S)$ *is not surjective.*

Proof. S is the integral closure of R in K (S) and is a finite R-module; hence, if $\mathfrak{m} \in \mathrm{Max}(R)$ and $\mathfrak{M}_1, \cdots, \mathfrak{M}_t$ are the maximal ideals of S over $\mathfrak{m}$, we get that

$$n = \sum_{1}^{t}{}_i\, e(\mathfrak{M}_i/\mathfrak{m}) f(\mathfrak{M}_i/\mathfrak{m}) = \sum_{1}^{t}{}_i\, e(\mathfrak{M}_i/\mathfrak{m}) > 1$$

(using [1], Chap 6, Th. 2. p. 147 and the hypotheses *b*), *c*)).

Since Div (R) and Div (S) are the free groups generated by the maximal ideals of R and S and since $i(\mathfrak{m}) = \sum_{1}^{t}{}_i\, e(\mathfrak{M}_i/\mathfrak{m}) \mathfrak{M}_i$, the conclusion is clear.

THEOREM. *Let* k *be an algebraically closed field and let* $(A, \mathfrak{m}, k)$ *be a local, 2-dimensional parafactorial ring such that at least one of the following conditions is satisfied*

i) char $(k) = 0$;

ii) A *contains a field* k_0 *and is a localization of a finitely generated* k_0-algebra.

Then A *is factorial.*

Proof. Let us suppose, by way of contradiction, that A is not factorial: then, using the notations and the conclusions of Lemma 1, we get that B and B$'$ are reduced and $\mathrm{Spec}(B') \to \mathrm{Spec}(B)$ is bijective; therefore we can write:

$$\text{in A} \quad \mathfrak{F} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r \qquad \mathfrak{p}_i \in \mathrm{Spec}(A) \qquad ht(\mathfrak{p}_i) = 1$$

$$\text{in A}' \quad \mathfrak{F} = \mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_r \qquad \mathfrak{P}_i \in \mathrm{Spec}(A') \qquad ht(\mathfrak{P}_i) = 1$$

where $\mathfrak{P}_i$ is the (unique) prime ideal of A$'$ lying over $\mathfrak{p}_i$. If we denote by $\mathfrak{p}$ one of the prime ideals $\mathfrak{p}_1, \cdots, \mathfrak{p}_r$ and by $\mathfrak{P}$ the corresponding prime ideal of A$'$, we get that $A_\mathfrak{p}$ is not integrally closed, since the conductor of $A_\mathfrak{p}$ in $(A_\mathfrak{p})' \simeq A'_\mathfrak{P}$ is $\mathfrak{F}A_\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$. Let us consider the canonical morphisms

$$\sigma : A_\mathfrak{p} \to A'_\mathfrak{P} \qquad \vartheta : A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \to A'_\mathfrak{P}/\mathfrak{P}A'_\mathfrak{P} .$$

We note that ϑ is not an isomorphism, otherwise $A'_\mathfrak{P} = \sigma(A_\mathfrak{p}) + \mathfrak{P}A'_\mathfrak{P}$, hence by Nakajama $A'_\mathfrak{P} = \sigma(A_\mathfrak{p})$, which contradicts the previous remark that $A_\mathfrak{p}$ is not integrally closed. Therefore, if we denote by K, K$'$ the fields of fractions of $A/\mathfrak{p}$, $A'/\mathfrak{P}$, it follows that $[K' : K] > 1$.

Our purpose is now to prove that the canonical morphism $\mathrm{Div}(\mathrm{B}) \to \mathrm{Div}(\mathrm{B}')$ is not surjective, which contradicts Lemma 1. As a first step we give a proof for the case in which $\mathfrak{s}$ is a prime ideal; for, let us denote by K, K' the fields of fractions of $\mathrm{B} = \mathrm{A}/\mathfrak{s}$, $\mathrm{B}' = \mathrm{A}'/\mathfrak{s}$, by $\overline{\mathrm{B}}, \overline{\mathrm{B}}'$ the integral closures of B, B' in K, K'. We have already proved that in this case $[\mathrm{K}' : \mathrm{K}] > 1$ and we note that for every maximal ideal $\mathfrak{m}$ of $\overline{\mathrm{B}}$, and for every maximal ideal $\mathfrak{M}$ of $\overline{\mathrm{B}}'$, $\overline{\mathrm{B}}/\mathfrak{m} \simeq \overline{\mathrm{B}}'/\mathfrak{M} \simeq k$ since k is algebraically closed. Moreover hypothesis i) (similarly hypothesis ii) implies that $\overline{\mathrm{B}}'$ is a finite $\overline{\mathrm{B}}$-module. Therefore we can apply Lemma 2 and we get that $\mathrm{Div}(\overline{\mathrm{B}}) \to \mathrm{Div}(\overline{\mathrm{B}}')$ is not surjective. Since the following natural commutative diagram

$$\begin{array}{ccc} \mathrm{Div}(\mathrm{B}) & \longrightarrow & \mathrm{Div}(\mathrm{B}') \\ \downarrow & & \downarrow \\ \mathrm{Div}(\overline{\mathrm{B}}) & \longrightarrow & \mathrm{Div}(\overline{\mathrm{B}}') \end{array}$$

has the vertical arrows surjective (see [2], 21.8.5.1. and 21.8.6.i), we are through.

As for the general case, let us denote by $\overline{\mathrm{B}}, \overline{\mathrm{B}}'$ the integral closures of B, B' in their total quotient rings; by B_i the ring $\mathrm{A}/\mathfrak{p}_i$, by B_i' the ring $\mathrm{A}'/\mathfrak{P}_i$ and by $\overline{\mathrm{B}}_i, \overline{\mathrm{B}}_i'$ the integral closures of $\mathrm{B}_i, \mathrm{B}_i'$ in their fields of fractions. Since $\overline{\mathrm{B}} \simeq \prod_i \overline{\mathrm{B}}_i$, $\overline{\mathrm{B}}' \simeq \prod_i \overline{\mathrm{B}}_i'$ (see [1], Chap. 5, Coroll. 1. p. 16), $\mathrm{Div}(\overline{\mathrm{B}}) \simeq \oplus\, \mathrm{Div}(\overline{\mathrm{B}}_i)$ and $\mathrm{Div}(\overline{\mathrm{B}}') \simeq \oplus\, \mathrm{Div}(\overline{\mathrm{B}}_i')$, the conclusion is now clear.

COROLLARY. *Let* $(\mathrm{X}, \mathcal{O}_{\mathrm{X}})$ *be an algebraic surface which is defined over an algebraically closed field and* $x \in \mathrm{X}$ *a closed point. Then every invertible sheaf on* $\mathrm{X} - \{x\}$ *can be uniquely extended to an invertible sheaf on* X *iff* $\mathcal{O}_{\mathrm{X},x}$ *is a factorial ring.*

Proof. It is a straightforward consequence of the previous theorem, after recalling the geometric meaning of parafactoriality (see [2], 21.13).

Remark 1. Let A be the ring which is obtained by localizing the ring $\mathbf{R}[\mathrm{X}, \mathrm{Y}, \mathrm{Z}]/(\mathrm{Y}^2 + \mathrm{X}^2 - \mathrm{X}^3)$ at the origin.

Using the notations of Lemma 2, an easy computation shows that $\mathrm{B} \simeq \mathbf{R}[\mathrm{V}]_{(\mathrm{V})}$ (V an indeterminate) $\mathrm{B}' \simeq \mathbf{C}[\mathrm{V}]_{(\mathrm{V})}$, $n = 2$, $f = 2$, hence $e = 1$ and $\mathrm{Div}(\mathrm{B}) \to \mathrm{Div}(\mathrm{B}')$ is an isomorphism.

Applying Lemma 1 we get that A is parafactorial and not factorial and this shows that the hypothesis "k algebraically closed" is essential in our theorem.

Remark 2. In [2], 21.13.9.1.vi p. 319 one can read that if k is an algebraically closed field and U, V, W are indeterminates, the ring A which is obtained by localizing at the origin the ring $k[\mathrm{U}, \mathrm{V}, \mathrm{W}]/(\mathrm{U}^2 - \mathrm{W}\mathrm{V}^2)$ is parafactorial. This is false because, after the theorem, the ring A, which is not normal, should be factorial; however a direct computation shows that $\mathrm{Div}(\mathrm{B}) \to \mathrm{Div}(\mathrm{B}')$ is not surjective.

Moreover, by directly calculating the Picard group of the punctured spectrum, it is possible to prove the following proposition:

Let k be a field; X , Y , Z indeterminates and $f(\mathrm{X}, \mathrm{Y}, \mathrm{Z})$ an irreducible polynomial of the type

$$\mathrm{X}^i \mathrm{Y}^j f_0(\mathrm{X}, \mathrm{Y}, \mathrm{Z}) - \mathrm{Z}^n f_1(\mathrm{X}, \mathrm{Y}, \mathrm{Z})$$

such that $f(0, 0, 0) = 0$, $i, j \geq 1$, $n \geq 2$, $f_0, f_1 \notin (\mathrm{X}, \mathrm{Y}, \mathrm{Z})$.

Then the ring which is obtained by localizing at the origin the ring $k[\mathrm{X}, \mathrm{Y}, \mathrm{Z}]/(f)$ is not parafactorial. Hence the fact that

$$k[\mathrm{U}, \mathrm{V}, \mathrm{W}]_{(\mathrm{U},\mathrm{V},\mathrm{W})}/(\mathrm{U}^2 - \mathrm{W}\mathrm{V}^2)$$

is not parafactorial is true and indipendent on the choice of the field k.

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