Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Anna Maria Micheletti, Francesco Zirilli

Nonlinear stability problems for a hyperbolic partial differential equation

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **59** (1975), n.6, p. 712–717.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1975_8_59_6_712_0>

Equazioni a derivate parziali. — *Nonlinear stability problems for a hyperbolic partial differential equation.* Nota di ANNA MARIA MICHELETTI e FRANCESCO ZIRILLI, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — In questa Nota preventiva si tratta della stabilità dinamica del problema 0.1, 0.2 (0.3), 0.4, 0.5 rispetto alle condizioni iniziali della soluzione banale e degli stati stazionari $u_n(x, \lambda)$. Si dànno condizioni necessarie e sufficienti per la stabilità dinamica.

INTRODUCTION

The equation

$$\frac{\partial^2 U}{\partial t^2} + \Gamma \frac{\partial U}{\partial t} - \Delta U - \lambda U + \beta U^3 = 0 \tag{0.0}$$

has been used to describe the dynamics of a variety of problems. It can be considered the time dependent Ginzburg-Landau equation in the presence of a dissipation term ($\Gamma > 0$) and in the absence of an electromagnetic field describing a superconductor [7]. Moreover the mechanical model of 0.0 in one space dimension is the following: a series of anharmonic oscillators with a common axis. The axis is fixed in space, the oscillator oscillates in parallel planes. Furthermore, the oscillators are interconnected by an axial torsion spring. The spring resists the relative rotation of neighboring oscillators. Finally, some friction force is active. We call this system continuous anharmonic oscillator.

We shall consider the linear dynamic stability of the static states of the equation 0.0; more precisely;

Let $\Omega \subset R^m$ be an open bounded connected set with $\partial\Omega$ enough regular.

We shall study 0.1, 0.2, 0.4, 0.5 (or 0.1, 0.3, 0.4, 0.4):

$$\frac{\partial U}{\partial t^2} - \Delta U + \Gamma \frac{\partial U}{\partial t} - \lambda U + \beta U^3 = 0 \qquad \text{on } \Omega \times [0 + \infty), \tag{0.1}$$

$$U = 0 \qquad \text{on } \partial\Omega \times [0 + \infty), \tag{0.2}$$

$$\frac{\partial U}{\partial \nu} + \gamma U = 0 \qquad \text{on } \partial\Omega \times [0 + \infty)\ \gamma > 0, \tag{0.3}$$

$$U(x, 0) = F(x) \qquad \text{on } \Omega, \tag{0.4}$$

$$\frac{\partial U}{\partial t}(x, 0) = G(x) \qquad \text{on } \Omega. \tag{0.5}$$

(*) Nella seduta dell'11 giugno 1975.

There exists an Ω^* arbitrarily close to Ω such that problem 0.6, 0.7 (or 0.6, 0.8).

$$-\Delta u - \lambda u + \beta u^3 = 0 \quad \text{on} \quad \Omega^*, \tag{0.6}$$

$$u = 0 \quad \text{on} \quad \partial\Omega^*, \tag{0.7}$$

$$\frac{\partial u}{\partial \nu} + \gamma u = 0 \quad \text{on} \quad \partial\Omega^* \quad \gamma > 0 \tag{0.8}$$

has infinitely many branches starting from the eigenvalues of the corresponding linearized problem. We call this solution static deflected states $u_n(x, \lambda)$. The $u_n(x, \lambda)$ are equilibrium states and they can be interpreted as excited states of the superconductor in the Ginzburg-Landau equation framework. In the following we give necessary and sufficient conditions in order to have the linear dynamic stability with respect to the initial data of the solution $u_n(x, \lambda)$ and $u \equiv 0$. Analogous work for the sine Gordon equation (Josephson junction) is contained in [1].

§ 1. THE ELLIPTIC PROBLEM

The equilibrium states (static states) are the time independent solutions $U(x, t) = u(x)$ of 0.1, 0.2, 0.4, 0.5 (0.1, 0.3, 0.4, 0.5), so that they satisfy the nonlinear elliptic problem 1.1, 1.2 (or 1.1, 1.3):

$$-\Delta u - \lambda u + \beta u^3 = 0 \quad \text{on} \quad \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on} \quad \partial\Omega, \tag{1.2}$$

$$\frac{\partial u}{\partial \nu} + \gamma u = 0 \quad \text{on} \quad \partial\Omega \quad \gamma > 0. \tag{1.3}$$

Since problem 1.1, 1.2 (1.1, 1.3) is well known, we merely summarize some of its properties.

THEOREM 1.1. *The first eigenvalue λ_0 of the the linearized problem corresponding to 1.1, 1.2 (or 1.1, 1.3) is a bifurcation point for 1.1, 1.2 (or 1.1, 1.3). The bifurcation set is a regular branch concave in a suitable left neighborhood of λ_0 and no bifurcation will occur for $\lambda < \lambda_0$.*

Proof. See reference [2]. Moreover:

THEOREM 1.2. *The branch starting at λ_0 of the nonlinear problem 1.1, 1.2 (1.1, 1.3) exists in $[\lambda_0, \lambda_c[$ and $\|u_\lambda\| \to +\infty$ as $\lambda \to \lambda_c$.*

Proof. See reference [6].

Problem 1.4, 1.5 (1.4, 1.6):

$$-\Delta u - \lambda u = 0 \quad \text{on} \quad \Omega, \tag{1.4}$$

$$u = 0 \quad \text{on} \quad \partial\Omega, \tag{1.5}$$

$$\frac{\partial u}{\partial \nu} + \gamma u = 0 \quad \text{on} \quad \partial\Omega \quad \gamma > 0, \tag{1.6}$$

where $\Omega \subset R^m$ is an open bounded set, may in general have multiple eigenvalues.

In the study if the bifurcation problem 1.2, 1.2 (1.1, 1.3) it is very useful to have only simple eigenvalues (odd multiplicity) [3].

So it is natural to ask if there exists a "little deformation" of Ω such that on the resulting Ω^* problem 1.4, 1.5 (1.4, 1.6) has only simple eigenvalues.

THEOREM 1.3. *Let $\Omega \subset R^m$ be an open bounded set of C^3 class. They in any ε-neighborhood of C^3 class there exists an open set Ω^* such that problem 1.4, 1.5 (1.4, 1.6) has only simple eigenvalues λ_i with eigenfunctions ψ_i.*

Proof. For problem 1.4, 1.5, see [4]. For problem 1.4, 1.6, see [5].

Let $C^3(R^m)$ be the Banach space of the continuous three times differentiable functions vanishing with the first three derivatives at infinity, equipped with the norm

$$|f|_{C^3} = \sup_x \max \{|f(x)|, |f'(x)| f''(x) |f'''(x)|\}.$$

DEFINITION. Let $\Omega \subset R^m$ be an open bounded set of Class C^3. We call ε-neighborhood of class C^3 of Ω the set

$$W_\varepsilon = \{\Omega' \subset R^m \mid \Omega' = (I + \psi)\cdot(\Omega) \,;\, \psi \in C^3(R^m) \,;\, |\psi|_{C^3} < \varepsilon\}$$

($I : R^m \to R^m$ is the identity).

THEOREM 1.4. *Problem 1.1, 1.2, (1.1, 1.3) on the set Ω^* has infinitely many bifurcation points $\lambda_n, \lambda_n \to +\infty$ corresponding to the eigenvalues of 1.4, 1.5 (1.4, 1.6) on Ω^*. The bifurcation sets are regular branch $u_n(x, \lambda)$.*

Proof. It is an immediate consequence of Theorem 1.3 and of standard results of bifurcation theory [3].

§ 2. THE STABILITY PROBLEM

We want to use the linear dynamic theory to test the stability of the trivial solution $u \equiv 0$ and of the static states $u_n(x; \lambda)$ for fixed λ (we write in the following $u_n(x)$ for $u_n(x, \lambda)$).

We consider the one parameter families of initial data $F(x, \eta)$ and $G(x, \eta)$ and the corresponding solutions $U(x, t, \eta)$ of 0.1, 0.2, 0.4, 0.5

(0.1, 0.3, 0.4, 0.5) such that

$$U(x;t;0) = u^m(x) \quad , \quad F(x,0) = u_n(x) \quad , \quad G(x;0) = 0.$$

The linear stability problem for $V(x,t) \equiv \frac{\partial U}{\partial \eta}(x,t,0)$ is the following:

$$V_{tt} - \Delta V + \Gamma V_t - \lambda_n u_n^2 V + \beta u_n^4 V = 0 \quad \text{on} \quad \Omega^* \times [0;+\infty[, \tag{2.1}$$

$$V = 0 \quad \text{on} \quad \partial\Omega^* \times [0;+\infty[, \tag{2.2}$$

$$\frac{\partial V}{\partial \nu} + \gamma V = 0 \quad \text{on} \quad \partial\Omega^* \times [0;+\infty[, \tag{2.3}$$

$$V(x;0) = \frac{\partial F}{\partial \eta}(x,0) \equiv F^0(x) \quad \text{on} \quad \Omega^*, \tag{2.4}$$

$$V_t(x,0) = \frac{\partial G}{\partial \eta}(x,0) \equiv G^0(x) \quad \text{on} \quad \Omega^*, \tag{2.5}$$

We first test the stability of the trivial solution $u \equiv 0$. Because $\{\psi_i\}$ is a complete orthonormal set of $L^2(\Omega^*)$ we assume that F^0, G^0 have a convergent expansion

$$F^0(x) = \Sigma F^k \psi_k(x) \quad , \quad G^0(x) = \Sigma G^k \psi_k(x).$$

The solution of 2.1, 2.2, 2.4, 2.5 (2.1, 2.3, 2.4, 2.5) with $u_n(x) \equiv 0$ and $\lambda_{(0)} = \lambda$ is:

$$V(xt) = \Sigma S_k(t)\, \psi_k \cdot (x)$$

where $S_k(t)$ are solutions of

$$S_k''(t) + \Gamma S_k'(t) + \lambda_k S_k = 0, \tag{2.6}$$

$$S_k(0) = F_k^0, \tag{2.7}$$

$$S_k'(0) = G_k^0. \tag{2.8}$$

Primes denote the differentiation with respect to t. An analysis of the solutions of 2.6, 2.7, 2.8 shows that the state $y \equiv 0$ is stable for any value of λ (Γ, λ_k are positive).

We now test the stability of the deflected static states $u_n(x)$. Consider the following eigenvalue problems:

$$-\Delta\varphi(x) - (\lambda_{(n)} u_n^2(x) - \beta u_n^4(x))\, \varphi(x) = \mu\varphi(x) \quad \text{on} \quad \Omega^*, \tag{2.9}$$

$$\varphi = 0 \quad \text{on} \quad \partial\Omega^*, \tag{2.10}$$

$$\frac{\partial \varphi}{\partial \nu} + \gamma\varphi = 0 \quad \text{on} \quad \partial\Omega^*. \tag{2.11}$$

From the regularity properties of $u_n(x)$ (they are critical points of a variational problem) it follows that problem 2.9, 2.10 (2.9, 2.11) has an infinite sequence of eigenvalues μ_j bounded below with eigenfunction φ_j such that $\lim_{j \to +\infty} \mu_j = +\infty$ and $+\infty$ is the only accumulation point of the sequence $\{\mu_j\}$.

We suppose now that the initial data F^0, G^0 have expansions in terms of $\{\varphi_j\}$:

$$F^0(x) = \Sigma \hat{F}_j^0 \varphi_j \quad , \quad G^0(x) = \Sigma \hat{G}_j^0 \varphi_j .$$

The solution V of 2.1, 2.2, 2.4, 2.5 (2.1, 2.3, 2.4, 2.5) will be of the following form:

$$V = \Sigma S_j(t) \varphi_j(x),$$

where S_j satisfies the following differential equation

$$S_j'' + \Gamma S_j' + \mu_j S_j = 0, \tag{2.12}$$

$$S_j'(0) = \hat{F}_j^0, \tag{2.13}$$

$$S_j'(0) = \hat{G}_0^j, \tag{2.14}$$

that is

$$S_j(t) = a_j^+ \exp p_j^+ t + a_j^- \exp p_j^- t,$$

where $p_j^{\pm}$ are the roots of

$$p^2 + \Gamma p + \mu_j = 0 \tag{2.15}$$

and $a_j^{\pm}$ are fixed by the initial conditions 2.13, 2.14.

THEOREM 2.1. *A static state $u_n(x;\lambda)$ is linearly dynamically stable for λ if and only if*

$$a_j^+ = 0 \qquad j = 0, 1, \cdots, m,$$

where $m = \max_n \{n \mid \mu_n \leq 0\}$. (Remarks: m is always finite).

Proof. The roots of 2.15 are

$$p_j^{\pm} = \frac{-\Gamma \pm \sqrt{\Gamma^2 - 4\mu_j}}{2} .$$

$\mathrm{Re}\, p_j^{\pm} \leq 0$ means linearly dynamically stable, when $\mathrm{Re}\, p_j^{\pm} > 0$ in order to have stability is necessary and sufficient $a_j^{\pm} = 0$ so we are done.

Concluding, the static deflected states $u_n(x, \lambda)$ such that the corresponding problem 2.9, 2.10 (2.9, 2.11) has negative eigenvalues are not linearly dynamically stable, for an arbitrary disturbance. However, following Theorem 2.1 they are stable for suitable restricted disturbances.

The static deflected states $u_n(x, \lambda)$ such that the corresponding problem 2.9, 2.10 (2.9, 2.11) has not negative eigenvalues and the trivial solution $u \equiv 0$ are linearly dynamically stable for an arbitrary disturbance.

REFERENCES

[1] CALLEGARI A. J. and REISS E. L. (1973) – *Nonlinear stability problems for the sine Gordon equation*, « J. Math. Phys. », *14*, 267–276.

[2] KELLER J. B. and ANTMAN S. (1969) – *Bifurcation theory and nonlinear eigenvalue problems*, Benjamin, New York.

[3] KRASNOSELSKI M. A. (1964) – *Topological methods in the theory of nonlinear integral equations*, MacMillan, New York.

[5] MICHELETTI A.M. (1972) – *Perturbazione dello spettro dell'operatore di Laplace in relazione a una variazione del campo*, « Ann. Scuola Norm. Sup. Pisa », *26*, 151–169.

[5] MICHELETTI A.M. – In preparazione.

[6] MICHELETTI A.M., PACE S. and ZIRILLI F. – *Some rigorous results about Ginzburg Landau equations*, to appear on « Nuovo Cimento B ».

[7] PARKS A. (1969) – *Superconductivity*, A.M. Dekker Inc., New York.