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Non-continuation of solutions of differential equations of order N

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Equazioni differenziali ordinarie. — *Non-continuation of solutions of differential equations of order* N. Nota di T. A. BURTON, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si dimostra un criterio sufficiente per la non prolungabilità delle soluzioni dell'equazione $x^{(n)} + a(t)f(x) = 0$ nelle ipotesi $a(t) < 0$, $xf(x) > 0$. Per $n = 3$ il criterio è parzialmente invertibile.

1. INTRODUCTION

In [1] Burton and Grimmer considered the equation

$$x'' + a(t)f(x) = 0, \qquad ' = d/dt, \tag{1}$$

in which a: $[0, \infty) \to (-\infty, \infty)$ and $f: (-\infty, \infty) \to (-\infty, \infty)$ with $xf(x) > 0$ if $x \neq 0$ and f and a continuous. It was shown that when $a(t_1) < 0$ for some t_1, then (1) has solutions satisfying $|x(t)| \to \infty$ as $t \to T^-$ for some $T > t_1$ if and only if either

$$\int_0^{\infty} [1 + F(x)]^{-1/2}\, dx < \infty \tag{a}$$

or

$$\int_0^{-\infty} [1 + F(x)]^{-1/2}\, dx > -\infty \tag{b}$$

holds where $F(x) = \int_0^x f(s)\, ds$.

The integrand in (a) and (b) turned out to be fundamental to second order equations. Petty and Johnson [6] generalized the result and used it to obtain a type of bound on solutions with explicit initial conditions. Komkov [5] extended the result to a more general second order equation. Burton and Grimmer showed that the necessity for non-continuation carried over to differential equations with a delay [3]. In other work ([2] and [3]) they showed that convergence of $\int_{0+}^{1} [F(x)]^{-1/2}\, dx$ relates to uniqueness of the zero solution.

(*) Nella seduta del 13 dicembre 1975.

In this Note we point out that the result of (a) and (b) can be generalized in one direction to an equation

$$x^{(n)} + a(t) f(x) = 0 \tag{2}$$

of arbitrary order. We obtain a partial converse when $n = 3$.

2. NON-CONTINUATION

In (2) let f and a satisfy the conditions in the sentence defining (1). We suppose that $a(t_1) < 0$ for some t_1 so that there exists $t_2 > t_1$ with $a(t) < 0$ on $[t_1, t_2]$. Also, there are positive numbers m and M with $-M \leq a(t) \leq -m$ on $[t_1, t_2]$. We define $F_n(x)$ by

$$F_1(x) = \int_0^x f(s)\, ds$$

and inductively,

$$F_{n+1}(x) = \int_0^x F_n(s)\, ds \qquad \text{for} \quad n = 1, 2, \cdots$$

THEOREM 1. *Let* $a(t) < 0$ *on* $[t_1, t_2]$. *If either*

$$\int_1^{\infty} [F_{n-1}(x)]^{-1/n}\, dx < \infty, \qquad \text{or} \tag{c}$$

$$\int_{-1}^{-\infty} [(-1)^n F_{n-1}(x)]^{-1/n}\, dx > -\infty \tag{d}$$

then (2) *has solutions* $x(t)$ *satisfying* $\lim_{t \to T^-} |x(t)| = +\infty$ *where* T *lies between* t_1 *and* t_2.

Proof. Suppose that (c) holds and let $x' = y$. Start a solution $x(t)$ of (2) at $t = t_1$ with $x(t_1) > 0, x'(t_1) > 0, \cdots, x^{(n-1)}(t_1) > 0$. In particular, it will turn out that the initial conditions should be determined as follows. Pick $x(t_1)$ so large that

$$\int_{x(t_1)}^{\infty} [mn\, F_{n-1}(s)]^{-1/n}\, ds < [t_2 - t_1]/2.$$

Then choose $y(t_1) > 0$ by $y^n(t_1) = mn\, F_{n-1}(x(t_1))$. Thus, with $x(t_1)$ and $y(t_1)$ fixed, define $y^{(n-2)}(t_1)$ by

$$y(t_1)\, y^{(n-2)}(t_1) = m\, F_1(x(t_1)),$$

and for $n > r + 2$ define

$$y^{r+1}(t_1)\, y^{(n-r-2)}(t_1) = m\, F_{r+1}(x)(t_1)).$$

These choices will occur naturally in the proof.

As $x^{(n)} = -a(t) f(x)$, so long as $x(t)$ is defined on $[t_1, t_2]$, say on $[t_1, T)$, we have $x(t) > 0, x'(t) > 0, \cdots, x^{(n-1)}(t) > 0$. Now $yx^{(n)} = -a(t) f(x) y \geq mf(x) y$ on $[t_1, T)$ so $yy^{(n-1)} \geq mf(x) x'$ and hence

$$\int_{t_1}^{t} y(s)\, y^{(n-1)}(s)\, ds \geq m\, [F_1(x)(t)) - F_1(x(t_1))]$$

from which an integration by parts yields

$$y(s)\, y^{(n-2)}(s) \Big|_{t_1}^{t} - \int_{t_1}^{t} y'(s)\, y^{(n-2)}(s)\, ds \geq m\, [F_1(x(t)) - F_1(x(t))].$$

Now the integral of $y'\, y^{(n-2)}$ in the last inequality is non-negative and so if we let $y(t_1)\, y^{(n-2)}(t_1) = m\, F_1(x(t_1))$, then we have

(3) $$y(t)\, y^{(n-2)}(t) \geq m\, F_1(x(t)).$$

In fact, using the above method one sees that if $r + 1 < n$ and

(4) $$y^r\, y^{(n-r-1)} \geq m\, F_r(x)$$

with $y^{r+1}(t_1)\, y^{(n-r-2)}(t_1) = m\, F_{r+1}(x(t_1))$, then

(5) $$y^{r+1}(t)\, y^{(n-r-2)}(t) \geq m\, F_{r+1}(x(t)).$$

To see this, multiply both sides of (4) by y and integrate both sides from t_1 to t obtaining

$$\int_{t_1}^{t} y^{r+1}(s)\, y^{(n-r-1)}(s)\, ds \geq m \int_{t_1}^{t} F_r(x(s))\, x'(s)\, ds.$$

Integrate the left side by parts obtaining

$$y^{r+1}(s)\, y^{(n-r-2)}(s) \Big|_{t_1}^{t} - \int_{t_1}^{t} (r+1)\, y^r(s)\, y'(s)\, y^{(n-r-2)}(s)\, ds \geq$$

$$\geq m\, [F_{r+1}(x(t)) - F_{r+1}(x(t_1))].$$

Using the initial condition and the fact that the above integrand is non-negative, we obtain immediately (5).

Thus, (3), (4), and (5) successively yield

$$y^2(t)\, y^{(n-1-2)}(t) \geq m\, F_2(x(t)),$$
$$y^3(t)\, y^{(n-2-2)}(t) \geq m\, F_3(x(t)),$$
$$\cdots\cdots\cdots\cdots$$
$$y^{n-2}(t)\, y'(t) \geq m\, F_{n-2}(x(t)).$$

(That is, note in (4) and (5) that the sum of the " exponents " of $y^r y^{(n-r-1)}$ is always $n - 1$ and we keep reducing to $n - r - 2 = 1$). If we multiply the last inequality by y and integrate from t_1 to t we obtain

$$y^n(t) - y^n(t_1) \geq mn\, [F_{n-1}(x(t)) - F_{n-1}(x(t_1))].$$

Define $y^n(t_1) = mn\, F_{n-1}(x(t_1))$ to obtain

(6) $$y(t)/[mn\, F_{n-1}(x(t))]^{1/n} \geq 1.$$

Integrate from t_1 to t obtaining

(7) $$\int_{x(t_1)}^{x(t)} [mn\; F_{n-1}(s)]^{-1/n}\, ds \geq t - t_1.$$

By choosing $x(t_1)$ so large that

(8) $$\int_{x(t_1)}^{\infty} [mn\, F_{n-1}(s)]^{-1/n}\, ds < [t_2 - t_1]/2,$$

we see in (7) that $x(t) \to \infty$ before t reaches t_2. As F_{n-1} is increasing, (*c*) implies that the integral in (8) converges so the choice for $x(t_1)$ is possible.

If (*d*) holds, then a similar proof is carried out with $x(t_1) < 0$, $x'(t_1) < 0, \cdots, x^{(n-1)}(t_1) < 0$ to complete the proof of the theorem.

When we attempt to carry out the proof of the converse theorem, we encounter certain difficulties and are forced to ask that $a(t) f(x)$ decreases when xx' is large. The following is the result for third order equations.

THEOREM 2. *In* (2) *let* $n = 3$. *If* $x(t_1) > 0$, $x'(t_1) > 0$, $x''(t_1) > 0$, *and* $a(t) < 0$ *on* $[t_1, t_2]$, *then* $x(t)$ *can be continued to* t_2 *provided that*

(*e*) $$\int_{1}^{\infty} [F_2(x)]^{-1/3}\, dx = +\infty$$

and when $x > 0$, $y > 0$, *and* $t \in [t_1, t_2]$, *then for large* x *and* y *we have*

(*f*) $$a'(t) f(x) + a(t) f'(x)\, y \leq 0.$$

Proof. As before, we have $-M \leq a(t) \leq -m < 0$ on $[t_1, t_2]$ and, so long as the solution is defined, say on $[t, T)$, we have $x' x''' = -a(t) f(x) x' \leq M f(x) x'$. If $x(T)$ is not finite, then we may rename t_1 so that we can assume x and y are so large on $[t_1, T)$ that (f) holds. Letting $x' = y$, we then obtain

$$\int_{t_1}^{t} y(s) y''(s) \, ds \leq M [F_1(x(t)) - F_1(x(t_1))].$$

But

$$\int_{t_1}^{t} y(s) y''(s) \, ds = y(t) y'(t) - y(t_1) y'(t_1) - \int_{t_1}^{t} [y'(s)]^2 \, ds$$

and so if we let $c_1 = y(t_1) y'(t_1) - MF_1(x(t_1))$, then we have $y(t) y'(t) \leq c_1 + \int_{t_1}^{t} [y'(s)]^2 \, ds + MF_1(x(t))$.

If we multiply by $y(t)$ and integrate from t_1 to t, then we obtain

$$(9) \qquad [y^3(t) - y^3(t_1)]/3 \leq c_1 [x(t) - x(t_1)] + M [F_2(x(t)) - F_2(x(t))] + \int_{t_1}^{t} y(s) \int_{t_1}^{s} [y'(u)]^2 \, du \, ds.$$

Under (e) we have $x^{(4)}(t) \geq 0$ and so an inequality of Fink [4] yields

$$(10) \qquad \int_{t_1}^{t} y(s) \int_{t_1}^{s} [y'(u)]^2 \, du \, ds \leq (2/9) y^3(t) + R(t)$$

where $R(t)$ is a polynomial in $y(t)$ of degree two or less. Thus, (9) and (10) yield

$$(11) \qquad (1/9) y^3(t) \leq c_1 [x(t) - x(t_1)] + M [F_2(x(t)) - F_2(x(t_1))] + \bar{R}(t)$$

where $\bar{R}(t)$ is a polynomial in $y(t)$ of degree ≤ 2. As we are assuming $y(t) \to \infty$, we may assume that t_1 is sufficiently close to T that (11) may be written as

$$(12) \qquad y^3(t) \leq L [x(t) + F_2(x(t)) + Q]$$

for appropriate positive constants L and Q. Thus, as $x'(t) = y(t)$ we obtain

$$(13) \qquad \int_{x(t_1)}^{x(t)} [L(s + F_2(s) + Q)]^{-1/3} \, ds \leq (t - t_1)$$

after taking roots of (12), integrating, and changing variable. By definition of $F_2(s)$, for large s we have $F_2(s) > [s+Q]\,k$ for some $k > 0$ and so (*e*) implies that the integral in (13) diverges as $x(t) \to \infty$. Thus, $x(t)$ is bounded on $[t_1, t_2]$. This completes a proof.

The counterpart of Theorem 2 for $x < 0$ is left to the reader. One asks that $\int_{-1}^{-\infty} [-F_2(x)]^{-1/3}\,dx = -\infty$ and works in the region in which $x < 0$, $x' < 0$, and $x'' < 0$.

The technique used here seems to fail for fourth and higher order equations.

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