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Oscillation of a forced nonlinear second order differential equation

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Equazioni differenziali ordinarie. — *Oscillation of a forced non-linear second order differential equation.* Nota di GARY D. JONES e SAMUEL M. RANKIN, III, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Gli Autori considerano l'equazione

(*) $$y'' + p(t) g(y) = f(t)$$

con $p(t)$, $f(t)$ continue in $[0, \infty)$, $p(t) = 0$, $g(t)$ continue in $(-\infty, \infty)$ e sotto opportune condizioni per $p(t)$, $g(y)$, $f(t)$ provano che tutte le soluzioni della (*) sono oscillanti.

This paper is concerned with the oscillatory character of the equation

(1) $$y'' + p(t) g(y) = f(t).$$

We will assume that p is positive and continuous on $[0, \infty)$, $f(t)$ is continuous on $[0, \infty)$ and g is continuous on $(-\infty, \infty)$.

Our results are closely related to those found in a paper by Atkinson [1]. However, our assumptions on p do not imply the oscillation of the unforced equation, as assumed in [1]. Other related results can be found in papers by Kartsatos [2, 3] and Kartsatos and Manougian [4], Rankin [5] and Teufel [6].

The assumptions that $xg(x) > 0$ for $x \neq 0$ and g is bounded away from zero when x is bounded away from zero, will be made throughout. By oscillation of a solution $y(t)$ of equation (1), is meant that $y(t)$ is continuable on (a, ∞) or some a and $y(t)$ has arbitrarily large zeros.

THEOREM 1. *If*

i) $$\overline{\lim_{t\to\infty}} \int_T^t (p(s) + \lambda f(s))\, ds = \infty \quad \textit{for each} \quad T > 0 \quad \textit{and for each } \lambda \neq 0$$

ii) *there exist points* $a, b > 0$ *such that*

$$h_1(t) \equiv \int_a^t (t - s) f(s)\, ds \geq 0 \qquad \textit{for all} \quad t \geq a;$$

$$h_2(t) \equiv \int_b^t (t - s) f(s)\, ds \leq 0 \qquad \textit{for all} \quad t \geq b.$$

(*) Nella seduta del 13 dicembre 1975.

iii) *$h_1(t)$ and $h_2(t)$ have arbitrarily large zeros.*

iv) $\left| \frac{1}{t} \int_T^t (t-s) f(s)\, ds \right| < M$ *for all* T

then every solution of equation (1) *oscillates.*

Proof. Suppose there exists a solution $y(t)$ of equation (1) that is non-oscillatory. We will assume $y(t) > 0$ on (c, ∞) for some $c > a$, $b > 0$, since $u = -y$ transforms (1) into an equation of the same form satisfying the assumption of the theorem.

We have that $(y(t) - h_1(t))'' + p(t) g(y) = 0$, which implies that $(y(t) - h_1(t))'' < 0$ for $t > c$. Thus $(y(t) - h_1(t))' > 0$ for all $t > c$, if this were not the case there would exist a $K > 0$ such that $y(t) - h_1(t) < -Kt$ for large t. This implies that

$$\frac{y(t)}{t} < -K + \frac{h_1(t)}{t}$$

for large t. Considering condition (iii) we have a contradiction to the positivity of $y(t)$.

Applying condition (iii) once more we see there exists a $T_1 > c$ such that $y(T_1) - h_1(T_1) > 0$. From $y'(t) > h_1'(t)$ we have by integrating from T_1 to t that

$$y(t) > y(T_1) + h_1(t) - h_1(T) \geq y(T_1) - h_1(T_1) = \beta > 0$$

for all $t > T_1$. Integrating equation (1) from T_2 to t for some $T_2 > T_1$ we get

$$y'(t) \leq y'(T_2) - \int_{T_2}^{t} (p(s) g(y(s)))\, ds < y'(T_2) - \beta \left(\int_{T_2}^{t} (p(s) - \frac{1}{\beta} f(s) \right) ds.$$

Condition (i) then implies that $\underline{\lim}_{t \to \infty} y'(t) = -\infty$.

Let $K_1 > 0$ be such that $M - K_1 < 0$. There exists a $T_3 > T_2$ such that $y'(T_3) < -K_1$. Integrating equation (1) twice from T_3 to t and dividing by t we have

$$\frac{y(t)}{t} \leq \frac{y(T_3)}{t} + \frac{y'(T_3)(t - T_3)}{t} + \frac{1}{t} \int_{T_3}^{t} (t-s) f(s)\, ds$$

$$< \frac{y(T_3)}{t} - \frac{K_1 (t - T_3)}{t} + M.$$

Thus $\overline{\lim}_{t \to \infty} \frac{y(t)}{t} \leq -K_1 + M < 0$. This again contradicts $y(t) > 0$ on (c, ∞).

Examples illustrating the above theorem are,

1) $y'' + t^{-3} y^3 = t \cos t$

2) $y'' + t^{-2} y^3 = t \cos t$

3) $y'' + (2 + \cos t) y = \sin \alpha t \, (\alpha > 0)$.

Example 1 does not satisfy the condition of Atkinson [1], while Examples 2 and 3 also satisfy the condition of Atkinson [1].

THEOREM 2. *If*

i) $\varliminf_{t\to\infty} \int_T^t f(s) \, ds = -\infty$ *and* $\varlimsup_{t\to\infty} \int_T^t f(s) \, ds = \infty$ *for all* $T > 0$,

ii) $\frac{1}{t} \left| \int_T^t (t-s) f(s) \, ds \right| < M$ *for all* $T > 0$

then every solution of equation (1) *is oscillatory.*

Proof. Suppose there exists an $a > 0$ such that $y(t) > 0$ on (a, ∞). From equation (1) we have

$$y'(t) \leq y'(T) + \int_T^t f(s) \, ds$$

for some $T > a$. Thus $\varliminf_{t\to\infty} y'(t) = -\infty$. Given $K > 0$ there exists a $T_1 > T$ such that $y'(T_1) < -K$. Choose K so that $M - K < 0$, then integrating (1) twice from T_1 to t and dividing by t we have

$$\frac{y(t)}{t} \leq \frac{y(T_1)}{t} + \frac{y'(T)(t-T_1)}{t} + \frac{1}{t} \int_{T_1}^t (t-s) f(s) \, ds$$

$$\leq \frac{y(T_1)}{t} + \frac{y'(T_1)(t-T_1)}{t} + M.$$

Therefore $\varlimsup_{t\to\infty} \frac{y(t)}{t} \leq y'(T) + M < -K + M < 0$. This contradicts $y(t) > 0$ on (a, ∞).

Theorem 2 can be applied to Examples 1 and 2, however it cannot be applied to Example 3. The equation

$$y'' + p(t) g(y) = (\sqrt{t} \sin t + k_1 t + k_2)'', \tag{4}$$

for some constants k_1, k_2 with p, g as above, has all solutions oscillatory by Theorem 2 but not by Theorem 1. It is interesting to note that allowing the forcing function to oscillate with unbounded amplitude reduces the need for "large" values of $p(t)$. The next theorem also illustrates this statement.

THEOREM 3. *If*

$$\lim_{t\to\infty} \frac{1}{t} \int_T^t (t-s) f(s)\, ds = -\infty$$

and

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \int_T^t (t-s) f(s)\, ds = \infty$$

for each $T > 0$ *then every solution* $y(t)$ *of equation* (1) *oscillates.*

Proof. Suppose $y(t)$ is a solution of equation (1) with $y(t) > 0$ on (a, ∞) for some $a > 0$. Integrating equation (1) twice from T to t for some $T > a$ and then dividing by t we have

$$\frac{y(t)}{t} \leq \frac{y(t)}{t} + \frac{y'(T)(t-T)}{t} + \frac{1}{t} \int_T^t (t-s) f(s)\, ds .$$

Under our hypothesis $\underline{\lim}_{t\to\infty} \frac{y(t)}{t} = -\infty$ which contradicts the positivity of $y(t)$.

Equations of the form

$$y'' + p(t) g(y) = e^t \sin t$$

with $p(t)$ and $g(y)$ as above satisfy the conditions of Theorem 3.

REFERENCES

[1] F. V. ATKINSON (1972/73) – *On second order differential inequalities*, « Proc. R.S.E.(A) », *72* (8), 109–127.

[2] A. G. KARTSATOS (1971) – *On the maintenance of oscillations of n^{th} order equations under the effect of a small forcing term*, « Journal of Differential Equations », *10*, 355–363.

[3] A. G. KARTSATOS (1972) – *Maintenance of oscillations under the effect of a periodic forcing term*, « Proc. Am. Math. Soc. », *33*, 377–383.

[4] A. G. KARTSATOS and M. N., MANOUGIAN (1974) – *Perturbations causing oscillations of functional-differential equations*, « Proc. Am. Math. Soc. », *43*, 111–117.

[5] S. M. RANKIN (1975) – *Oscillation of a forced second order non-linear differential equation.* « Proc. Am. Math. Soc. », *58*, July, 1976.

[6] H. JR. TEUFEL (1972) – *Forced second-order non-linear oscillations*, « J. Math Anal. Applic. », *40*, 148–152.