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John R. Graef, Paul W. Spikes

Nonoscillation theorems for forced second order non linear differential equations

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Equazioni differenziali ordinarie. — *Nonoscillation theorems for forced second order nonlinear differential equations* (*). Nota di JOHN R. GRAEF e PAUL W. SPIKES, presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — Gli Autori provano alcuni nuovi criteri sufficienti, indipendenti da altri criteri da loro ottenuti in precedenza, perché gli integrali dell'equazione

$$(a(t)\,x')' + q(t)\,f(x)\,g(x') = r(t)$$

siano tutti non oscillatori.

1. INTRODUCTION

Since 1955 when the well known paper by Atkinson [1] was published, a great deal of work has been done on the oscillatory behavior of second order nonlinear differential equations of the type

$$(a(t)\,x')' + q(t)\,f(x)\,g(x') = r(t) \tag{*}$$

particularly when $r(t) \equiv 0$. Excellent surveys of known results can be found in Wong [25, 26] and Coffman and Wong [2, 3]. Many results guaranteeing that equation (*) be oscillatory can be found in the literature, but there are relatively few which guarantee that (*) has a nonoscillatory solution. In the latter case we refer the reader to the discussion in [2] and the results of Izyumova and Kiguradze [11], Moore and Nehari [16], Onose [19–21], Wong [24–26], and Yoshizawa [27]. Even fewer conditions guaranteeing that (*) be nonoscillatory are known (see [2, 3, 4, 10, 12, 14, 17, 18]), and except for some results on linear equations [8, 13, 22], the authors know of no such results when $r(t) \not\equiv 0$ other than those in [5–7].

The main results in this paper are contained in Theorems 3 and 4. They give sufficient conditions for all solutions of (*) to be nonoscillatory without assuming that $r(t) \equiv 0$.

The last section of this paper contains some examples which show that the results obtained here are independent of each other as well as from those in [5–7].

2. NONOSCILLATION THEOREMS

The proof of some of the results in this paper depend upon the following "comparison" type theorem for linear equations. Consider the equations

$$(a(t)\,x')' + h(t)\,x = r(t) \tag{1}$$

and

$$(a(t)\,x')' + k(t)\,x = 0 \tag{2}$$

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where $a, h, k, r: [t_0, \infty) \to \mathrm{R}$, a, k and r are continuous, $a(t) > 0$, and $r(t)$ does not vanish identically on any subinterval of $[t_0, \infty)$. (Note that we have not required h to be continuous).

We classify solutions of (1) (and of equation (3) below) as follows. A solution $x(t)$ will be called nonoscillatory if there exists $t_1 \geq t_0$ such that $x(t) \neq 0$ for $t \geq t_1$; the solution will be called oscillatory if for every $t_1 \geq t_0$ there exist t_2 and t_3 with $t_1 < t_2 < t_3$, $x(t_2) > 0$, and $x(t_3) < 0$; and it will be called a Z-type solution if it has arbitrarily large zeros but is ultimately nonnegative or nonpositive.

THEOREM 1. *Suppose that* $h(t) \leq k(t)$ *and* (2) *is nonoscillatory.*

i) *If* $r(t) \geq 0$ $(r(t) \leq 0)$, *then no solution of* (1) *is oscillatory or nonnegative* (*nonpositive*) *Z-type.*

If, in addition, either

ii) $h(t) \geq 0$,

or,

iii) $h(t) \equiv k(t)$,

then equation (1) *is nonoscillatory.*

Proof. Let $r(t) \geq 0$, $y(t)$ be an oscillatory or nonnegative Z-type solution of (1), and $u(t)$ be a solution of (2). Defining $\mathrm{H}(t) = a(t)[u(t)y'(t) - y(t)u'(t)]$, we have $\mathrm{H}'(t) = u(t)r(t) + [k(t) - h(t)]y(t)u(t)$. Since (2) is nonoscillatory we may assume that $u(t)$ is ultimately positive. Hence there exist t_1 and t_2 with $t_2 > t_1 > t_0$ such that $y(t_1) = y(t_2) = 0$, $y(t) > 0$ for t in (t_1, t_2), and $u(t) > 0$ for $t \geq t_1$. It then follows that $0 \leq \mathrm{H}(t_1) < \mathrm{H}(t_2) \leq 0$ which is impossible. A similar argument for the case when $r(t) \leq 0$ completes the proof of i).

To prove ii) in case $r(t) \geq 0$, it suffices to show that (1) has no nonpositive Z-type solutions. If $y(t)$ is such a solution, there exists $t_3 \geq t_0$ such that $y(t) \leq 0$ for $t \geq t_3$ and $y'(t_3) = 0$. We have $(a(t)y'(t))' = r(t) - h(t)y(t) \geq 0$ for $t \geq t_3$ and integrating we obtain $a(t)y'(t) \geq 0$ for $t \geq t_3$. This implies that $y'(t) \geq 0$ for $t \geq t_3$ which is impossible for a Z-type solution. Similarly, if $r(t) \leq 0$, then (1) has no nonnegative Z-type solutions.

In order to prove iii) for $r(t) \geq 0$, assume that $u(t)$ is an ultimately positive solution of (2) and $y(t)$ is a nonpositive Z-type solution of (1), say $u(t) > 0$ and $y(t) \leq 0$ for $t \geq t_4 \geq t_0$. There exist t_5 and t_6 satisfying $t_6 > t_5 > t_4$, $y(t_5) = y(t_6) = 0$, and $y(t) < 0$ for t in (t_5, t_6). Now $\mathrm{H}(t_5) < \mathrm{H}(t_6)$ but since $y(t)$ is a Z-type solution, $y'(t_5) = y'(t_6) = 0$, so $\mathrm{H}(t_5) = \mathrm{H}(t_6) = 0$ and we have a contradiction. A similar argument holds if $r(t) \leq 0$.

Remark. Part iii) of Theorem 1 generalizes a result of Švec [22], Hammett [8], and Keener [13].

Now consider the equation

$$(a(t)x')' + q(t)f(x)g(x') = r(t) \tag{3}$$

where $a, q, r : [t_0, \infty) \to \mathrm{R}$ and $f, g : \mathrm{R} \to \mathrm{R}$ are continuous, $a(t) > 0$, $g(y) > 0$, and $r(t)$ does not vanish identically on any subinterval of $[t_0, \infty)$. Let $q'(t)_+ = \max\{q'(t), 0\}$ and $q'(t)_- = \max\{-q'(t), 0\}$ so that $q'(t) = q'(t)_+ - q'(t)_-$. Define $\mathrm{F}(x) = \int_0^x f(s)\,\mathrm{d}s$, $\mathrm{G}(y) = \int_0^y [s/g(s)]\,\mathrm{d}s$ and assume that

$$q(t) > 0, \tag{4}$$

$$\mathrm{F}(x) > -\mathrm{D} \quad \text{for some} \quad \mathrm{D} > 0, \tag{5}$$

$$\int_{t_0}^{\infty} [q'(s)_+/q(s)]\,\mathrm{d}s < \infty, \tag{6}$$

$$\int_{t_0}^{\infty} [a'(s)_-/a(s)]\,\mathrm{d}s < \infty, \tag{7}$$

$$\int_{t_0}^{\infty} [|r(s)|/a(s)]\,\mathrm{d}s < \infty, \tag{8}$$

$$\mathrm{G}(y) \to \infty \quad \text{as} \quad |y| \to \infty, \tag{9}$$

and there are positive constants M and d such that

$$y^2/g(y) \leq \mathrm{MG}(y) \quad \text{for} \quad |y| \geq d. \tag{10}$$

The following lemma will be used in the proof of Theorem 3.

LEMMA 2. *Suppose conditions* (4)–(10) *hold. If $x(t)$ is a solution of* (3), *then $x'(t)$ is bounded and there exists $\mathrm{A} > 0$ and $\mathrm{T} \geq t_0$ such that $|x(t)| \leq \mathrm{A}t$ for $t \geq \mathrm{T}$.*

Proof. Write equation (3) as the system

$$x' = z$$

$$z' = (-a'(t)\,z - q(t)\,f(x)\,g(z) + r(t))/a(t)$$

and define

$$\mathrm{V}(x, z, t) = \mathrm{G}(z) + q(t)\,[\mathrm{F}(x) + \mathrm{D}]/a(t).$$

Then

$$\begin{aligned} \mathrm{V}' = &- a'(t)\,z^2/a(t)\,g(z) + r(t)\,z/a(t)\,g(z) + \\ &+ [\mathrm{F}(x) + \mathrm{D}]\,(a(t)\,q'(t) - q(t)\,a'(t))/a^2(t) \leq a'(t)_-\,z^2/a(t)\,g(z) + \\ &+ r(t)\,z/a(t)\,g(z) + [\mathrm{F}(x) + \mathrm{D}]\,q'(t)_+/a(t) + \\ &+ [\mathrm{F}(x) + \mathrm{D}]\,q(t)\,a'(t)_-/a^2(t) \leq a'(t)_-\,z^2/a(t)\,g(z) + \\ &+ r(t)\,z/a(t)\,g(z) + (q'(t)_+/q(t) + a'(t)_-/a(t))\,\mathrm{V}. \end{aligned}$$

From (10) it follows that $z^2/g(z) \leq M_1 + MG(z)$ for all z and some $M_1 > 0$. If $|z| \leq 1$, then $|z|/g(z)$ is bounded, and if $|z| \geq 1$, then $|z|/g(z) \leq z^2/g(z)$ so there exists $M_2 > 0$ such that $|z|/g(z) \leq M_2 + MG(z)$ for all z. Thus

$$V' \leq MG(z)\, a'(t)_-/a(t) + M_1 a'(t)_-/a(t) + MG(z)\,|r(t)|/a(t) +$$
$$+ M_2 |r(t)|/a(t) + (q'(t)_+/q(t) + a'(t)_-/a(t))\, V \leq M_1\, a'(t)_-/a(t) +$$
$$+ M_2 |r(t)|/a(t) + [q'(t)_+/q(t) + (M+1)\, a'(t)_-/a(t) + M|r(t)|/a(t)]\, V.$$

Integrating, we obtain

$$V(t) \leq V(t_0) + \int_{t_0}^{t} [M_1\, a'(s)_-/a(s) + M_2 |r(s)|/a(s)]\, ds +$$
$$+ \int_{t_0}^{t} [q'(s)_+/q(s) + (M+1)\, a'(s)_-/a(s) + M\,|r(s)|/a(s)]\, V(s)\, ds\,.$$

Now the first integral above is bounded so

$$V(t) \leq M_3 + \int_{t_0}^{t} [q'(s)_+/q(s) + (M+1)\, a'(s)_-/a(s) + M\,|r(s)|/a(s)]\, V(s)\, ds$$

and an application of Gronwall's inequality shows that V is bounded for all $t > t_0$. Therefore $G(z(t))$ is bounded and hence $z(t) = x'(t)$ is bounded for all $t \geq t_0$, say $|x'(t)| \leq B$. Thus $|x(t)| \leq |x(t_0)| + B(t - t_0)$ and so there exists $A > 0$ and $T \geq t_0$ such that $|x(t)| \leq At$ for $t \geq T$.

Remark. The above lemma generalizes a result in [5]. Notice that the restriction that $r(t)$ does not vanish on any subinterval of $[t_0, \infty)$ is not needed here. Also, if $r(t) \equiv 0$ and $a'(t) \geq 0$, then condition (10) can be dropped. Finally, note that by condition (8), if $a(t)$ is large, then the forcing term $r(t)$ may be large.

From the proof of Lemma 2 it is easy to see that if conditions (4), (5), (9), and (10) hold, then solutions of (3) are defined for all $t \geq t_0$. In what follows and without further mention, we will assume that solutions of (3) are continuable.

We will make use of the following additional assumptions on equation (3). Assume that there is a continuous function W such that

(11) $$|f(x)| \leq |x|\, W(x) \quad \text{for } x \neq 0\,,$$

(12) $$W(x_1) \leq W(x_2) \quad \text{if } |x_1| \leq x_2\,,$$

(13) $$f(0) = 0\,.$$

We will also need the following condition.

Condition N. For any choice of the constants $K > 0$ and $c > 0$ equation (2) with $k(t) = Kq(t) W(ct)$ is nonoscillatory.

Although Condition N may seem somewhat formidable, it is relatively easy to verify whether or not it holds by checking known nonoscillation results for linear equations. For example,

$$\text{if} \quad a(t) \equiv 1 \qquad \text{and} \quad \int_{t_0}^{\infty} sq(s) W(cs) \, ds < \infty$$

(Hille [9]), or if

$$\int_{t_0}^{\infty} [1/a(s)] \, ds < \infty \qquad \text{and} \quad \int_{t_0}^{\infty} q(s) W(cs) \, ds < \infty$$

(Moore [15]), then Condition N would hold. For other such possibilities we refer the reader to Swanson [23] and the references contained therein.

THEOREM 3. *Suppose that* (4)-(13) *and Condition* N *hold. If* $r(t) \geq 0$ ($r(t) \leq 0$), *then no solution of* (3) *is oscillatory or nonnegative* (*nonpositive*) Z-*type. If, in addition, either*

$$(14) \qquad r(t) \neq 0 \quad on \quad [t_0, \infty),$$

$$(15) \qquad xf(x) \geq 0 \quad for\ all \quad x,$$

or,

$$(16) \qquad f(x)/x \to L \quad as \quad x \to 0,$$

then equation (3) *is nonoscillatory*.

Proof. Let $x(t)$ be a solution of (3) and define $h : [t_0, \infty) \to R$ by $h(t) = q(t) f(x(t)) g(x'(t))/x(t)$ if $x(t) \neq 0$, and $h(t) = 0$ if $x(t) = 0$. Then we see that $x(t)$ is a solution of (1). By Lemma 2, $x'(t)$ is bounded so $g(x'(t))$ is bounded, say $|g(x'(t))| \leq K$ for $t \geq t_0$. Also there exist $A > 0$ and $T \geq t_0$ such that $|x(t)| \leq At$ for $t \geq T$, so $W(|x(t)|) \leq W(At)$ for $t \geq T$. Hence $h(t) \leq Kq(t) W(At) = k(t)$ so by Condition N and part i) of Theorem 1 the proof of the first part of this theorem is complete.

If (14) holds, say $r(t) > 0$ for $t \geq t_0$, it suffices to show that (3) has no nonpositive Z-type solutions. Suppose $x(t)$ is such a solution and let $t_1 \geq t_0$ be a zero of $x(t)$. Then $x'(t_1) = 0$, and from (3) and (13) we have $a(t_1) x''(t_1) = r(t_1) > 0$. This implies that $x''(t_1) > 0$ which is impossible since $x(t)$ attains a relative maximum at $t = t_1$.

If, instead, (15) holds, then the function $h(t)$ defined above satisfies $h(t) \geq 0$ and so part ii) of Theorem 1 applies.

Finally, if (16) holds, then define h by $h(t) = q(t) f(x(t)) g(x'(t))/x(t)$ if $x(t) \neq 0$, and $h(t) = Lq(t) g(x'(t))$ if $x(t) = 0$. Thus h is continuous and $x(t)$ is a solution of (1). First observe that $f(x)/x \leq |f(x)|/|x| \leq W(x)$ if $x \neq 0$, so $L \leq W(0)$. As before, there exist $K > 0$, $A > 0$ and $T \geq t_0$ such that $|x(t)| \leq At$ for $t \geq T$ and $|g(x'(t))| \leq K$ for all $t \geq t_0$. Hence if $x(t) \neq 0$, $h(t) \leq Kq(t)W(At)$ and if $x(t) = 0$, $h(t) \leq Kq(t)W(0)$ so $h(t) \leq Kq(t)W(At)$ for all $t \geq T$. By Condition N and the Sturm comparison theorem, equation (2) with $h(t) \equiv k(t)$ is nonoscillatory. Thus by part iii) of Theorem 1, equation (1) is nonoscillatory, so $x(t)$ is nonoscillatory.

Remark. If $a(t) \equiv 1$, $\int_{t_0}^{\infty} sq(s)W(cs)\,ds < \infty$ for all $c > 0$, and either (14) or (15) holds, then Theorem 3 extends a result of Kamenev [12] to forced equations.

The condition that $r(t)$ does not change signs is essential. For example, the equation

$$x'' + x^3/t^5 = (\sin^3(\ln t) - t^3 \sin(\ln t) - t^3 \cos(\ln t))/t^5, \qquad t \geq 1$$

satisfies the hypotheses of Theorem 3 but $x(t) = \sin(\ln t)$ is an oscillatory solution. An example of a linear equation with the same properties can be found in [6].

By placing stronger conditions on the functions W and g we can obtain a result similar to Theorem 3 but with much weaker hypotheses on a, q and r.

THEOREM 4. *Suppose conditions* (11) *and* (13) *hold,*

(17) $$W(x) \leq P \quad \textit{for all } x,$$

(18) $$g(y) \leq K \quad \textit{for all } y,$$

and the equation

(19) $$(a(t)x')' + PK\,|q(t)|\,x = 0$$

is nonoscillatory. If $r(t) \geq 0$ $(r(t) \leq 0)$, *then no solution of* (3) *is oscillatory or nonnegative* (*nonpositive*) *Z-type. If, in addition, either*

i) *condition* (14) *holds,*

ii) $q(t) \geq 0$ *and condition* (15) *holds,*

or,

iii) *condition* (16) *holds and* $L \leq P$,

then equation (3) *is nonoscillatory.*

Proof. Letting $x(t)$ be a solution of (3) and defining $h(t)$ as in the first part of the proof of Theorem 3, we have $h(t) \leq PK\,|q(t)|$. Since equation (19) is nonoscillatory, we can invoke part i) of Theorem 1 to obtain the first conclusion. The remainder of the proof is similar to the proof of the corresponding parts of Theorem 3 and will be omitted.

Remark. Since we did not invoke Lemma 2 in the proof of Theorem 4, the restrictions on a, q and r have been relaxed considerably. Except in part ii) we do not need a sign condition on $q(t)$, we do not have $a(t)$ bounded from below which is implied by condition (7), and no restriction is placed on the size of $r(t)$ even if $a(t)$ is bounded from above. In addition, $F(x)$ need not be bounded from below. Finally, note that equation (19) may be conditionally oscillatory which is not true for equation (2) under Condition N.

Remark. We can drop conditions (17) and (18) from the hypotheses of Theorem 4 if we consider only bounded solutions of (3) whose derivatives are also bounded. In this case the constants in equation (19) would depend on the choice of the solution.

3. Examples

In this section we present some examples which show that the different nonoscillation results in the previous section are independent of each other as well as from those obtained by the authors in [5–7].

Example 1. Consider the equation

$$x'' + f(x)/t^5 = r(t), \qquad t \geq 1$$

where $f(x) = x \ln(|x| + 2) \cos(1/x)$ if $x \neq 0$, and $f(0) = 0$. Here we can let $W(x) = |x| + 2$ and it is clear that Condition N holds. If $r(t) = (1 + \sin t)/t^2$, then the covering hypotheses of Theorem 3 are satisfied but we cannot conclude nonoscillation. If, instead, $r(t) = 1/t^2$, then condition (14) is satisfied so the equation is nonoscillatory. Notice that neither (15) nor (16) holds.

Example 2. If in the equation

$$x'' + f(x)/t^5 = (1 + \sin t)/t^2, \qquad t \geq 1$$

$f(x) = x \ln(|x| + 2) |\cos(1/x)|$ if $x \neq 0$ and $f(0) = 0$, then (15) is satisfied but (14) and (16) are not. On the other hand, if $f(x) = x^3 \cos(1/x)$ if $x \neq 0$, $f(0) = 0$, then (16) holds but (14) and (15) do not.

Notice that in all of the cases described in the first two examples, $W(x)$ was not bounded so Theorem 4 did not apply.

Example 3. For the equation

$$(t^2 x')' + f(x)/t^2 = r(t), \qquad t \geq 1$$

with $f(x) = x \sin(1/x)$ if $x \neq 0$, $f(0) = 0$, and $r(t) = t^3 (1 + \cos t)$, the hypotheses of Theorem 4 are satisfied since the linear equation

$$(t^2 x')' + x/t^2 = 0, \qquad t \geq 1$$

is nonoscillatory by the criteria of Moore [15] cited above. Theorem 3 does not apply since condition (8) fails to be satisfied. This is also the case if $r(t) = t^3$, and in this case the above equation would be nonoscillatory. Neither of the other parts of Theorem 4 hold for this choice of $f(x)$. Now by letting $r(t) = t^3 (1 + \cos t)$ and either (*a*) $f(x) = x \mid \sin (1/x) \mid$ if $x \neq 0$ and $f(0) = 0$, or (*b*) $f(x) = x \sin x$, we can deduce that the various parts of Theorem 4 are independent of each other.

Remark. None of the examples described in this section satisfy any of the nonoscillation results in [5–7].

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