Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Javed Ahsan

A note on rings all of whose semi-simple cyclic modules are quasi-injective

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **59** (1975), n.6, p. 682–684.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1975_8_59_6_682_0>

Analisi funzionale. — *A note on rings all of whose semi-simple cyclic modules are quasi-injective.* Nota (*) di JAVED AHSAN, presentata dal Corrisp. A. ANDREOTTI.

RIASSUNTO. — Si studiano gli anelli soddisfacenti alla proprietà indicata nel titolo e si mostra che, nel caso commutativo, tale proprietà caratterizza gli anelli che – modulo il radicale – risultano artiniani e semisemplici.

It is well-known that rings all of whose modules are injective are semi-simple artinian. Osofsky [8] proved that rings over which every cyclic module is injective are also semi-simple artinian. In [2], Cateforis and Sandomierski proved, in the commutative case, that a ring is semi-simple artinian even if only its semi-simple (cyclic) modules are assumed to be injective. Later, Michler and Villamayor [7] proved that this characterization of semi-simple artinian rings remains valid also in the general case. As regards the question of classifying rings with similar conditions on modules in the quasi-injective setting we recall that rings all of whose modules are quasi-injective are semi-simple artinian (see [4], Cor. 2.4). Rings for which every cyclic module is quasi-injective have also been studied (see eg. [1]). The purpose of this brief note is to study rings all of whose semi-simple cyclic modules are quasi-injective. We shall prove, in the commutative case, that this property is characterized by the fact that, modulo their radical, such rings are semi-simple artinian. Before we prove this result, some preliminary definitions are included.

A module M over a ring R will be called semi-simple if the (Jacobson) radical of M is zero, i.e. if the intersection of all maximal submodules of M is zero. Socle of a module is defined to be the sum of all its simple submodules. A module M is called finite dimensional (Goldie) if there do not exist infinitely many non-zero submodules whose sum is direct. Generalizing the notion of injective modules, Johnson and Wong [5] called an R-module M 'quasi-injective' if every homomorphism from a submodule of M to M can be extended to an endomorphism of M. For various properties of these modules we refer to [5] and [3]. Simple modules are trivially quasi-injective. Also, semi-simple artinian modules are quasi-injective. A ring R is called self-injective if R_R is an injective module. R is called regular in the sense of Von Neumann if $a \in aRa$, for each $a \in R$. R is said to be a local ring if R has a unique maximal (right) ideal. A ring R is called an FGS-ring if each cyclic R-module over this ring has finitely generated (or empty) Socle. Kurshan [6] proved that a ring R is FGS if and only if each finitely generated R-module is finite dimensional. R is said to be a qc-ring if each cyclic R-module is quasi-injective (see [1]).

(*) Pervenuta all'Accademia l'11 agosto 1975.

Throughout this note we shall assume that rings are commutative and have the identity element and all modules are unitary. For an R-module M, J (M) will denote the Jacobson-radical of M and J = J (R) will denote the Jacobson-radical of the ring R.

We start with the following definition. A ring R will be called a *generalized qc–ring* if each semi-simple cyclic R-module is quasi-injective. Clearly, every qc–ring is generalized qc. However, generalized qc-rings need not be qc. We support this statement with the following proposition.

PROPOSITION 1. *Let* R *be a local ring. Then* R *is a generalized qc-ring.*

Proof. Let M_R be any cyclic R-module which is also semi-simple. Suppose $M_R = R/I$; I an ideal of R. Since J (R/I) = o, it follows that $J \subseteq I$ but J is the unique maximal ideal of R; therefore J = I. This implies that R/I is a simple R-module and so quasi-injective. Thus R is a generalized qc-ring.

We next prove the following lemma.

LEMMA. *Let* R *be any ring. Then* R *is generalized qc* $\langle = \rangle$ *each factor ring of* R *is so.*

Prof. Suppose R is generalized qc-ring. Let $\bar{R} = R/I$ be a factor ring of R, where I is some ideal of R. Consider a semi-simple cyclic $\bar{R}$-module $M_{\bar{R}}$. Clearly, M_R is also a semi-simple cyclic R-module. Since R is generalized qc, M_R is quasi-injective. This implies that $M_{\bar{R}}$ is quasi-injective (see Lemma 2 of [1]). Therefore $\bar{R}$ is a generalized qc–ring.

We now prove the main proposition of this note.

PROPOSITION 2. *Let* R *be any commutative ring. Then the following statements are equivalent*:

(1) R *is generalized qc*;

(2) R/J *is semi-simple artinian.*

Proof. (1) Suppose that R is a generalized qc-ring. Since R/J is a semi-simple cyclic R-module, R/J is (R —) quasi-injective. This implies that $\bar{R} = R/J$ is a self-injective ring. Further, it follows from a result of Faith and Utumi [4] (see Lemma 1 of [1]) that $\bar{R}$ is regular in the sense of Von Neumann. Also, in view of the above lemma, $\bar{R}$ is a generalized qc-ring. Since $\bar{R}$ is regular in the sense of Von-Neumann, each cyclic $\bar{R}$-module is semi-simple by Theorm 4 of [2]. Therefore, each cyclic $\bar{R}$-module is ($\bar{R}$ —) quasi-injective. This means that $\bar{R}$ is a qc-ring. Therefore by the corollary on P. 428 of [1], $\bar{R}$ is semi-simple artinian.

(2) Now, suppose that R/J is semi-simple artinian. We prove that R is a generalized qc-ring. Let M_R be any cyclic R-module which is also semi-simple. Suppose $M_R \cong R/I$; I an ideal of R. Since J (R/I) = o, it follows that $J \subseteq I$ so that $I/J \subseteq R/J$ (as an ideal). Since $R/J/I/J \cong R/I$;

R/I, being a homomorphic image of a semi-simple artinian ring, is a semi-simple artinian ring. This implies that R/I is (R/I —) quasi-injective. Therefore R/I is (R —) quasi-injective. This proves the proposition.

Finally, we employ a standard argument to prove the following proposition. But first we remark that if $0 \to A \to B \to C \to 0$ is any short exact sequence of R-modules with A and C finite demensional then it is a known result (see eg. Kurshan [6]) that M is also finite dimensional.

PROPOSITION 3 *Let* R *be a generalized commutative qc-ring. Then R is an FGS-ring.*

Proof. The proposition will follow if we prove that each finitely generated R-module is finite dimensional. We use inductive argument to prove this fact. Let M_R be a module generated by one element. Then M_R is a cyclic R-module, say, $M_R \simeq R/I$ (I an ideal of R). Now $(R/I)_R$ is finite dimensional if and only if R/I/J (R/I) is so. Since R/I is a generalized qc-ring, R/I/J (R/I) is semi-simple artinian by Proposition 2 and hence finite dimensional. This implies that $(R/I)_R$ is finite dimensional. Let us now assume that the result is true for $n = k$ and show that the result is true also for $n = k + 1$. Let us write $M = Rx, + \cdots + Rx_{k+1}$. Suppose $A = Rx_1$ and $B = M/A$. Then B is finite dimensional by the inductive hypothesis. Also, since A is a cyclic R-module, A is finite dimensional. Now consider the exact sequence $0 \to A \to M \to B \to 0$. Since A and B are finite dimensional, it follows from the above remark, that M is also finite dimensional. This completes the proof of the proposition.

Acknowledgements. The author would like to acknowledge conversations with Professors C. Traverso and A. Andreotti. He also wishes to thank professors Abdus Salam and P. Budini as well as the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

REFERENCES

[1] J. AHSAN (1973) – *Rings all of whose cyclic modules are quasi-injective,* « Proc. London Math. Soc. », *27* (3), 425–439.

[2] V. C. CATEFORIS and F. L. SANDOMIERSKI (1969) – *On commutative rings over which the singular submodule is a direct summand for every module,* « Pac. J. Math. », *31*, 289–292.

[3] C. FAITH (1967) – *Lectures on injective modules and quotient rings,* Springer–Verlag, Berlin.

[4] C. FAITH and Y. UTUMI (1964) – *Quasi-injective modules and their endomorphism rings,* « Arch. Math. », *15*, 166–174.

[5] R. E. JOHNSON and E. T. WONG (1961) – *Quasi-injective modules and irreducible rings,* « J. London Math. Soc. », *36*, 260–268.

[6] R. P. KURSHAN (1970) – *Rings whose cyclic modules have finitely generated socle,* « J. Algebra », *15*, 370–386.

[7] G. O. MICHLER and O. E. VILLAMAYOR (1973) – *On rings whose simple modules are injective,* « J. Algebra », *25*, 185–201.

[8] B. L. OSOFSKY (1968) – *Non–injective cyclic modules,* « Proc. Amer. Math. Soc. » *19*, 1383–1384.