Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

A. G. A. G. Babiker

Some measure theoretic properties of completely regular spaces. Nota II

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **59** (1975), n.6, p. 677–681.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1975_8_59_6_677_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1975.

Analisi matematica. — *Some measure theoretic properties of completely regular spaces.* Nota II (*) di A. G. A. G. BABIKER, presentata (**) dal Socio B. SEGRE.

RIASSUNTO. — Ved. la Nota I qui citata in calce.

§ 3. THE TOPOLOGY σ AND SECOND CHARACTERIZATION OF ESSENTIALLY LINDELÖF SPACES

Under the uniform norm topology, $C^*(X)$ is isometrically isomorphic to $C(\beta X)$. So that, if X is not compact, the dual of $C^*(X)$ cannot be identified with the set of all signed measures on X. Furthermore, the topology of X cannot be recaptured from the Banach space structure of $C^*(X)$. Since our purpose is to use the topological linear structure of $C^*(X)$ to characterize those spaces in which every Baire measure is net-additive, the uniform norm topology is not adequate. So, we first define a locally convex topology σ, on $C^*(X)$, giving as dual the set of all signed Baire measures on X, and determining the topology of X uniquely for a wide class of completely regular spaces.

Write

$$\mathscr{H} = \{h \in C^* : 0 < h \leq 1\}.$$

For $h \in \mathscr{H}$, let s_h be the topology on C^* defined by the norm $\| \ \|_h$, where $\|f\|_h = \|fh\| = \sup_{x \in X} |f(x)\, h(x)|$.

Define σ_h to be the finest locally convex topology which agrees with s_h on uniformly bounded sets, and let

$$\sigma = \inf \{\sigma_h : h \in \mathscr{H}\},$$

where the infimum is taken in the lattice of locally convex topologies on $C^*(X)$.

The following theorem follows from various results established in [3]. For completeness, we give a more direct proof.

THEOREM 3.1. *The dual of* (C^*, σ) *is the set of all norm bounded* σ*-additive linear functionals on* C^*, *so that it can be identified with the set of all signed Baire measures on* X.

(*) Continuation of « Nota I », appeared in the same volume of these « Rendiconti », p. 362.

(**) Nella seduta del 13 dicembre 1975.

Proof. Let L be a linear functional on C^* which is continuous with respect to σ. Since σ is clearly coarser than the uniform norm topology, L is norm bounded. Thus $\exists$ a finitely additive set function μ defined on the algebra generated by all the zero sets of X such that

$$L(f) = \int f d\mu, \qquad \text{for all } f \in C^*.$$

Let $\{Z_n\}$ be a sequence of zero sets such that:

(i) $Z_n \nearrow X$, and,

(ii) for each n $\exists$ a positive set P_n such that $Z_n \subset P_n \subset Z_{n+1}$.

Such a sequence is called a *regular sequence* [14, p. 168] (1). It follows from [14, Th. 13] that, for some $f \in C^*$, we have:

$$Z_n = \left\{x : h(x) \geq \frac{1}{n}\right\}.$$

Clearly, $h \in \mathscr{H}$. Let $\varepsilon > 0$ be given. Since L is σ_h-continuous, $\exists \eta > 0$ such that $\left|\int f d\mu\right| < \varepsilon$ whenever $\|fh\| < \eta$, and $\|f \leq 1\|$. For any $n > 1/\eta$, and any $f \in C^*$ such that $\|f\| \leq 1$ and $f(Z_n) = 0$, we have $\left|\int f d\mu\right| < \varepsilon$. It follows that $|\mu|(Z) < \varepsilon$ for any zero set Z such that $Z \cap Z_n = \varnothing$. Hence $|\mu|(X \setminus Z_n) < \varepsilon$. i.e. $|\mu|(X \setminus Z_n) \to 0$. It follows from [14, th. 19] that μ is σ-additive. So L is σ-additive.

Conversely, suppose that L is a σ-additive linear functional on C^*. Let μ be the corresponding signed Baire measure on X. To show that L is continuous with respect to σ, it is sufficient to prove that L is s_h-continuous on the unit ball for all $h \in \mathscr{H}$.

Let $h \in \mathscr{H}$ be given. For each positive integer, let,

$$Z_n = \left\{x : h(x) \geq \frac{1}{n}\right\}:$$

clearly $\{Z_n\}$ is a regular sequence. Since μ is a signed Baire measure, $|\mu|(X \setminus Z_n) \to 0$ as $n \to \infty$.

Let $\varepsilon > 0$ be given. $\exists m$ such that $|\mu|(X \setminus Z_m) < \varepsilon/2$. The set:

$$V = \left\{f \in C^* : |f| \leq 1 \quad \text{and } \|fh\| < \frac{\varepsilon}{2m\|\mu\|}\right\}$$

is an s_h-neighbourhood of O in the unit ball. We want to show that $|L(f)| < \varepsilon$, for all $f \in V$.

(1) Referred to literature given at the end of Part I of this Note (Nota I).

If $f \in \mathrm{V}$, then:

$$|\mathrm{L}(f)| = \left|\int_{\mathrm{X}} f d\mu\right| \leq \int_{\mathrm{X}} |f| \, d|\mu| = \int_{\mathrm{Z}_m} |f| \, d|\mu| + \int_{\mathrm{X}\setminus \mathrm{Z}_m} |f| \, d|\mu| < \frac{\varepsilon}{1} + \frac{\varepsilon}{2} = \varepsilon .$$

So L is σ-continuous and the proof is complete.

We know that the topology σ is coarser than the uniform norm topology. In the following theorem we characterize those spaces for which the two topologies agree.

THEOREM 3.2. X *is pseudocompact if and only if* σ *agrees with the uniform norm topology on* $\mathrm{C}^*(\mathrm{X})$.

This follows from the fact, shown in [3], that σ is strongly Mackey. Here we give a more direct proof.

Proof. Suppose that X is pseudocompact. To show that σ agrees with the norm topology, it is sufficient to prove that the unit ball $\mathrm{B} = \{f \in \mathrm{C}^* : \|f\| \leq 1\}$ is a σ-neighbourhood of O. For each $h \in \mathscr{H}$, let $\alpha_h = \inf\{h(x) ; x \in \mathrm{X}\}$. Since $1/h$ is a continuous function on X and so bounded, $\alpha_h > 0$. The set $\mathrm{V} = \{f \in \mathrm{C}^* : \|fh\| < \alpha_h\}$ is an s_h-neighbourhood of O which is contained in B. So B is an s_h-neighbourhood of O, for all $h \in \mathscr{H}$, and so a σ-neighbourhood of O.

The converse follows from (3.1) and [10, Th. 3.1], and the proof is complete.

Theorem (3.1) implies that the multiplicative linear functionals in the dual of (C^*, σ) are precisely those induced by two-valued Baire measures on X. When X is realcompact, these measures are unit-point-masses on X. This establishes a bijection between X and the set $\mathscr{M}$ of all σ-closed maximal ideals in $\mathrm{C}^*(\mathrm{X})$. $\mathscr{M}$, endowed with the Stone topology [7, p. 58], is clearly homeomorphic to X. Therefore: Two realcompact spaces X and Y are homeomorphic if and only if $(\mathrm{C}^*(\mathrm{X}), \sigma)$ and $(\mathrm{C}^*(\mathrm{Y}), \sigma)$ are isomorphic. This generalizes Gelfand-Kolmogoroff theorem [7, p. 57].

Using the algebraic and topological structure of (C^*, σ), we give the following characterization of essentially Lindelöf spaces.

THEOREM 3.3. *A completely regular space* X *is essentially Lindelöf if, and only if, every closed ideal in* $(\mathrm{C}^*(\mathrm{X}), \sigma)$ *is fixed.*

For the proof, we need

LEMMA 3.4. *Pointwise multiplication of functions in* C^* *is separately continuous with respect to* σ.

Proof. First, note that a σ-neighbourhood base of O may be chosen to consist of the family of convex hulls of sets of the form:

$$\bigcup_{h \in \mathscr{H}} \bigcup_{n=1}^{\infty} \{f \in \mathrm{C}^* : \|f\| \leq n \quad \text{and} \quad \|fh\| < c_{n,h}\}$$

where, for each $h \in \mathscr{H}$, $\{c_{n,h}\}$ is a sequence of positive numbers which may be chosen to be monotonically decreasing to 0.

Let $g \in C^*$, and let V be a convex σ-neighbourhood of O. For each $h \in \mathscr{H}$, $\exists$ a sequence $\{c_{n,h}\}$ of positive numbers such that:

$$\bigcup_{n=1}^{\infty} \{f : \|f\| \leq n \|g\| \quad \text{and} \quad \|fh\| < c_{n,h}\} \subset V .$$

Let,

$$W = \bigcup_{h \in \mathscr{H}} \bigcup_{n=1}^{\infty} \{f : \|f\| \leq n \quad \text{and} \quad \|fh\| < \frac{c_{n,h}}{\|g\|}\} .$$

Clearly $U = \operatorname{conv}(W)$, the convex hull of W, is a σ-neighbourhood of O, and $gU \subset V$. It follows that, for each $g \in C^*$, the map: $f \to gf$ is σ-continuous.

Proof of Theorem 3.3. Suppose that X is essentially Lindelöf and let I be a σ-closed ideal in $C^*(X)$. By the Hahn-Banach theorem, $\exists$ a σ-continuous linear functional L such that $L(I) = 0$ and $L \not\equiv 0$. It follows from (3.1) and (2.1) that $L^{-1}(0)$ is a sequential hyperplane. As $I \subset L^{-1}(0)$, (2.2) implies that I is fixed.

Conversely, suppose that X is not essentially Lindelöf. By (2.2) $\exists$ a sequential hyperplane H containing a free ideal I. By (2.1) and (3.1), $H = L^{-1}(0)$ for some σ-continuous linear functional L. Hence H is σ-closed, and so $\bar{I}$, the closure of I with respect to σ, is contained in H. It follows from (3.4) that $\bar{I}$ is an ideal which is clearly free. This completes the proof.

§ 4. Locally compact spaces

Since locally compact spaces are open in their Stone-Čech compactifications, the properties of being essentially Lindelöf and that of essential compactness are equivalent for such spaces. Suppose that X is locally compact and let $C_*(X) \subset C^*(X)$ be the algebra of all functions admitting a continuous extension to the one-point compactification of X, i.e. $f \in C_*(X)$ if and only if $f \in C^*(X)$ and $\exists$ a real number r_f such that, for any $\varepsilon > 0$, $\exists$ a compact set $K \subset X$ such that $|f(x) - r_f| < \varepsilon$ for all $x \in X \setminus K$. Define L on $C_*(X)$ by:

$$L(f) = r_f .$$

L is a positive linear functional on $C_*(X)$. It induces a set function μ_0 on a sub-σ-algebra of the Baire sets of X. This sub-σ-algebra is the σ-algebra generated by those zero subsets of X which are either relatively compact or have relatively compact complements. By the Hahn-Banach theorem, L can be extended to a linear functional $\hat{L}$ on $C^*(X)$. Any such an extension $\hat{L}$ which satisfies $\|\hat{L}\| = \|L\|$ is necessarily a positive functional, and hence induces a positive finitely addition set function μ defined on all the Baire subsets of X, which extends μ_0. The following proposition is immediate.

PROPOSITION 4.1. *Let* X *be locally compact and let* L *be defined on* $C_*(X)$ *as above. Then* X *is essentially compact if and only if any extension of* L *to a norm bounded linear functional on* $C^*(X)$ *is purely finitely additive.*

(We recall that a positive linear functional is purely finitely additive if it has no σ-additive minorant).

Next, we use the topology σ to characterize locally compact essentially compact spaces. Let $C_k(X)$ denote the set of all continuous functions on X with compact support. Whether X is locally compact or not, $C_k(X)$ is the intersection of all the free ideals in $C^*(X)$, and $C_k(X)$ is itself a free ideal if, and only if, X is locally compact but not compact [7, p. 61].

THEOREM 4.2. *A locally compact space* X *is essentially campact if and only if* $C_k(X)$ *is* σ*-dense in* $C^*(X)$.

Proof. Suppose that $C_k(X)$ is not σ-dense. It follows from Lemma (3.4) that $\overline{C_k(X)}$, the σ-closure of $C_k(X)$, is an ideal in $C^*(X)$. Since X is locally compact, $C_k(X)$ is a free ideal. Thus $\overline{C_k(X)}$ is a σ-closed free ideal. It follows from (3.3) that X is not essentially compact.

Conversely, suppose that X is not essentially compact. By (3.3) there exists a σ-closed free ideal I. Therefore $C_k(X) \subset I$. As I is σ-closed, $C_k(X)$ is not σ-dense. This completes the proof.

Local compactness in (4.2) is crucial. The one-point Lindelöfization of an uncountable discrete space with non-measurable cardinal is clearly essentially compact. Let L be the linear functional on $C^*(X)$ whose associated Baire measure is the unit-point-mass at infinity. Clearly L is σ-continuous and $C_k(X) \subset L^{-1}(0)$, so that $C_k(X)$ is not σ-dense.

Finally, we settle a question raised by Kirk [9] by showing that a locally compact realcompact space need not be essentially compact. The following example have been used for different purposes in [4] and [5].

Example. Let I be the closed unit interval [0,1] and let X be the subset of $\mathbf{R}^2$ consisting of $D=\{0\} \times I$ and the points $\left\{\left(\frac{1}{n}, \frac{k}{n}\right)\right\} \cap I \times I$, n, k positive integers. For each $(0, y) \in D$ and each positive integer m, let

$$T_m(y) = \left\{(u, v) \in X : u \leq \frac{1}{m} \quad \text{and} \quad |v - y| \leq u\right\},$$

Take $\{T_m(y)\}$, $m = 1, 2, \cdots$ as a neighbourhood base for $(0, y) \in D$; and for $x \in X \setminus D$, let $\{x\}$ be a neighbourhood of x.

It is easy to verify that, for each m and each y, $T_m(y)$ is a compact metric space. Hence X is locally compact and locally metrizable.

Since the natural injection $i : X \to \mathbf{R}^2$ is continuous and since every subspace of $\mathbf{R}^2$ is realcompact, it follows from [6, Th. 8. 18] that every subspace of X is realcompact, and so X is realcompact. An argument used by Moran in [11, p. 637] can be used to show that every Baire set in X is a Baire set in $\mathbf{R}^2$. So, the linear Lebesgue measure on D induces a Baire measure on X which is clearly without support. Therefore X is not essentially compact.