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Model-completion in categories

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Algebra. — *Model–completion in categories.* Nota di GEORGE GEORGESCU e IONA PETRESCU, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Per una categoria vengono introdotti gli analoghí dei concetti di « model–companion » e « model–completion » dovuti al Robinson, ottenendo poi varie conseguenze ed applicazioni ad essi relative.

We define, for a category, analogues of Robinson's concepts of model-companion and model-completion of a theory and obtain in this context some results corresponding to those in [1], [4], [7]. The categorical aspect can serve to obtain the model-companion or the model-completion of certain categories of models which admit a functor with a right adjoint into a category of models with known model-companion or model-completion.

1. Let $\mathscr{C}$ be a category closed under directed limits and direct products.

Let I be a set and $\mathscr{D}$ be an ultrafilter over I. If A is an object in $\mathscr{C}$, we define, as in [8], the ultrapower of A over $\mathscr{D}$ to be the directed limit $\varinjlim_{F \in \mathscr{D}} A^F$. Obviously, $\mathscr{C}$ is closed under ultrapowers.

The following two definitions are motivated by the Keisler-Shelah theorem.

DEFINITION 1. Two objects A, B in $\mathscr{C}$ are called elementarily equivalent (denoted $A \equiv B$) if they have isomorphic iterated ultrapowers, i.e. if there are pairs $(I_1, \mathscr{D}_1), \cdots, (I_n, \mathscr{D}_n)$ and $(J_1, \mathscr{E}_1), \cdots, (J_m, \mathscr{E}_m)$ where $\mathscr{D}_i$, $\mathscr{E}_j$ are ultrafilters over I_i, J_j respectively $(i = 1, \cdots, n)$, $(j = 1, \cdots, m)$ such that:

$$(\cdots (A^{I_1}/\mathscr{D}_1)^{I_2}/\mathscr{D}_2 \cdots)^{I_n}/\mathscr{D}_n \simeq (\cdots (B^{J_1}/\mathscr{E}_1)^{J_2}/\mathscr{E}_2 \cdots)^{J_m}/\mathscr{E}_m .$$

DEFINITION 2. A monomorphism $A \to B$ is called elementary if there are isomorphic iterated ultrapowers A′, B′ of A, B such that the following diagram is commutative:

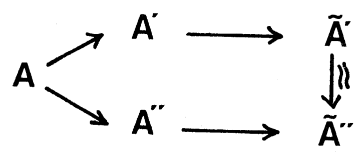
$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \xrightarrow{\approx} & B' \end{array}$$

where the vertical morphisms are the canonical ones.

(*) Nella seduta del 13 dicembre 1975.

In what follows we shall always suppose that $\mathscr{C}$ has the following property:

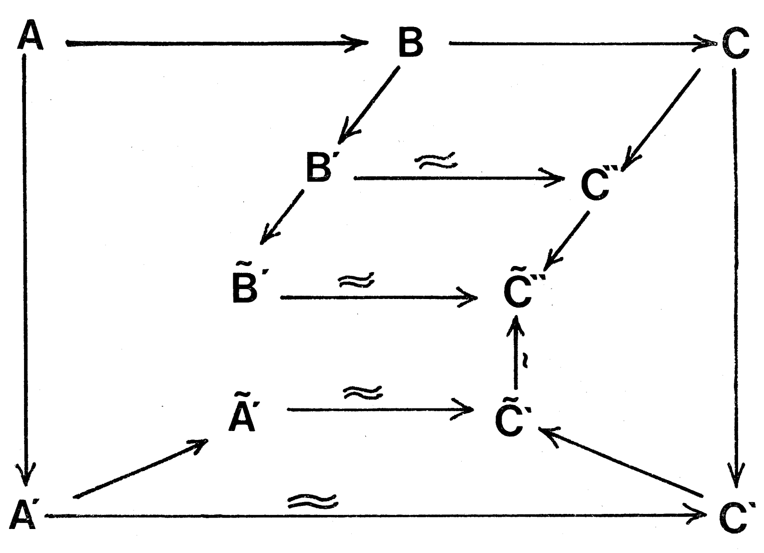
($\equiv$) any two iterated ultrapowers A', A'' of an object A are elementarily equivalent over A, that is they have isomorphic iterated ultrapowers $\tilde{A}', \tilde{A}''$ such that the following diagram commutes:



(here and throughout this paper, the morphisms of an object into an ultrapower of it will be supposed to be the canonical ones).

LEMMA 1. *If the morphisms* $A \to B \to C$ *and* $B \to C$ *are elementary, then so is* $A \to B$.

Proof. Using Definition 1 and property ($\equiv$) we obtain the following commutative diagram:



where A', B', C', C'' are iterated ultrapowers of A, B, C respectively and $\tilde{C}', \tilde{C}''$ are iterated ultrapowers of C', C'', $\tilde{B}'$ is obtained from B' by the same ultrapower operations as $\tilde{C}''$ from C'' and $\tilde{A}'$ is obtained similarily from A. Then $\tilde{A}', \tilde{B}'$ are the desired isomorphic iterated ultrapowers of A, B.

2. DEFINITION 3. Let $\mathscr{C}^*$ be a full subcategory of $\mathscr{C}$. $\mathscr{C}^*$ is called a model-companion of $\mathscr{C}$ if:

M 1) for any object A in $\mathscr{C}$ there exists a monomorphism $A \to B$ with B in $\mathscr{C}^*$ ($\mathscr{C}^*$ is model-consistent with $\mathscr{C}$);

M 2) every monomorphism in $\mathscr{C}^*$ is elementary ($\mathscr{C}^*$ is model-complete);

M 3) $\mathscr{C}^*$ is a maximal full subcategory of $\mathscr{C}$ with properties M1, M2.

PROPOSITION 1. *Let $\mathscr{C}^*$ be a model-complete full subcategory of $\mathscr{C}$. The following are equivalent*:

(i) *$\mathscr{C}^*$ is a model-companion of $\mathscr{C}$;*

(ii) *any object* A *in $\mathscr{C}$ satisfying the property that every monomorphism* $A \to B$ *with* B *in $\mathscr{C}^*$ is elementary, is itself in $\mathscr{C}^*$.*

Proof. (i) ⇒ (ii). Let A be an object with the property in (ii). We show that the full subcategory $\mathscr{C}^* \cup \{A\}$ has properties M1, M2, thus from M3 it follows that A is in $\mathscr{C}^*$. If $A \to C$ is a monomorphism with C in $\mathscr{C}^*$ then, from (ii) it is elementary. Let $C \to A$ be a monomorphism with C in $\mathscr{C}^*$; there exists a morphism $A \to A^*$ with A^* in $\mathscr{C}^*$. Then $C \to A \to A^*$ and $A \to A^*$ are elementary and from Lemma 1, $C \to A$ is also. So $\mathscr{C}^* \cup \{A\}$ is model-complete. Its model-consistency with $\mathscr{C}$ is obvious.

(ii)⇒(i). If A is an object in $\mathscr{C}$ such that there is no monomorphism $A \to A^*$ with A^* in $\mathscr{C}^*$, then A satisfies trivially the condition in (ii), hence is in $\mathscr{C}^*$. So $\mathscr{C}^*$ satisfies M1.

Let $\mathscr{C}'$ be a full subcategory of $\mathscr{C}$ satisfiying M1, M2 and containing $\mathscr{C}^*$. Let A be an object in $\mathscr{C}'$ and $A \to A^*$ a monomorphism with A^* in $\mathscr{C}^*$. By properties M1, M2 of $\mathscr{C}'$ we obtain elementary morphisms $A^* \to A^{**}$ and $A \to A^* \to A^{**}$ with A^{**} in $\mathscr{C}'$, and using Lemma 1 again, $A \to A^*$ is elementary. Then, by (ii) A is in $\mathscr{C}^*$ and the maximality of $\mathscr{C}^*$ follows. Hence, $\mathscr{C}^*$ is a model-companion of $\mathscr{C}$.

We shall say that a category $\mathscr{C}$ has the Tarski-Vaught property if for every chain of elementary morphisms in $\mathscr{C}$, its limit is an elementary extension of every object in the chain.

LEMMA 2. *Let $\mathscr{C}$ be a category with the Tarski-Vaught property and $\mathscr{C}_1$, $\mathscr{C}_2$ be full subcategories of $\mathscr{C}$ both satisfying* M1, M2. *Then their union has the same properties.*

Proof. Let $A_1 \to A_2$ be a monomorphism with A_1 in $\mathscr{C}_1$ and A_2 in $\mathscr{C}_2$. Using M1 we obtain an alternating chain:

$$A_1 \to A_2 \to \cdots \to A_n \to \cdots$$

where A_{2n+1} is in $\mathscr{C}_1$ and A_{2n} is in $\mathscr{C}_2$ for every $n \geq 1$. By M2 the chains $(A_{2n})_{n\geq 1}$, $(A_{2n+1})_{n\geq 0}$ are elementary and obviously they have common limit. By the Tarski-Vaught property and Lemma 1, $A_1 \to A_2$ is elementary. Hence, the union of $\mathscr{C}_1$ and $\mathscr{C}_2$ has property M2. Property M_1 is obviously satisfied.

REMARK. From Lemma 2 and M3 it follows that if a category with the Tarski-Vaught property has a model-companion, this is unique.

In the remainder of this paper we always suppose that $\mathscr{C}$ is a category having the Tarski-Vaught property.

LEMMA 3. *If $\mathscr{C}$ contains a full subcategory $\mathscr{C}'$ model-complete and model-consistent with $\mathscr{C}$ then $\mathscr{C}$ has a model-companion.*

Proof. The result is immediate by taking $\mathscr{C}^*$ to be the union of model-complete and model-consistent with $\mathscr{C}$, full subcategories of $\mathscr{C}$ and using Lemma 2.

THEOREM 1. *Let $\mathscr{C}$, $\mathscr{C}'$ be categories closed under directed limits and direct products and satisfying the Tarski-Vaught property and property* ($\equiv$). *Let* $T : \mathscr{C} \rightarrow \mathscr{C}'$ *be a faithful functor and* $S : \mathscr{C}' \rightarrow \mathscr{C}$ *a full faithful functor right adjoint to* T *and ultrapowers preserving. If $\mathscr{C}'$ has a model-companion $\mathscr{C}'^*$ then* $S(\mathscr{C}'^*)$ *is model-complete and model-consistent with $\mathscr{C}$, so $\mathscr{C}$ has a model-companion.*

Proof. Let $\Psi : 1_{\mathscr{C}} \rightarrow S \circ T$ and $\Phi : T \circ S \rightarrow 1_{\mathscr{C}'}$ be the adjointness morphisms where Φ is an isomorphism and Ψ is a monomorphism. If A is in $\mathscr{C}$, there is a monomorphism $TA \rightarrow C$ in $\mathscr{C}'$ with C in $\mathscr{C}'^*$ and $A \xrightarrow{\Psi_A} STA \rightarrow SC$ gives a monomorphism into an object of $S(\mathscr{C}'^*)$. Hence $S(\mathscr{C}'^*)$ has property M1.

Let $SA \xrightarrow{Sf} SB$ be a monomorphism in $S(\mathscr{C}'^*)$. By M2, $A \xrightarrow{f} B$ is elementary so there exist isomorphic iterated ultrapowers A′, B′ of A, B respectively, such that the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ \downarrow & & \downarrow \\ A' & \xrightarrow{\quad \approx \quad} & B' \end{array}$$

Then, the commutative diagram

$$\begin{array}{ccc} S(A) & \xrightarrow{\quad S(f) \quad} & S(B) \\ \downarrow & & \downarrow \\ S(A') & \xrightarrow{\quad \approx \quad} & S(B') \end{array}$$

and the fact that S preserves ultrapowers gives M2 for $S(\mathscr{C}'^*)$.

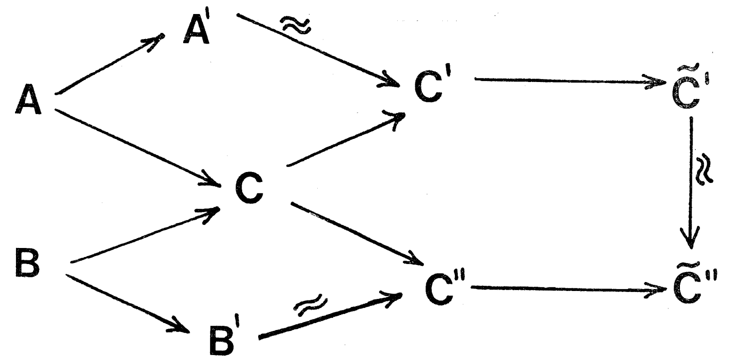
THEOREM 2. *Let $\mathscr{C}$ be a category having a model-companion $\mathscr{C}^*$. The following statements are equivalent*:

(i) *any two objects in $\mathscr{C}^*$ are elementarily equivalent*;

(ii) *for any two objects* A, B *in $\mathscr{C}$ there are monomorphisms* $A \rightarrow C$ $B \rightarrow C$ *in $\mathscr{C}$ ($\mathscr{C}$ has the joint embedding property).*

Proof. (i) $\Rightarrow$ (ii). If A, B are objects in $\mathscr{C}$ there exist monomorphisms $A \rightarrow A^*$, $B \rightarrow B^*$ with A^*, B^* in $\mathscr{C}^*$. By (i), A^*, B^* are elementarily equivalent hence, using Definition 1, we obtain (ii).

(ii) $\Rightarrow$ (i). Let A, B be objects in $\mathscr{C}^*$. By (ii) and M1 there exist monomorphisms $A \rightarrow C$, $B \rightarrow C$ with C in $\mathscr{C}^*$. By M2, these are elementary, so we obtain, using also property ($\equiv$) the following diagram:

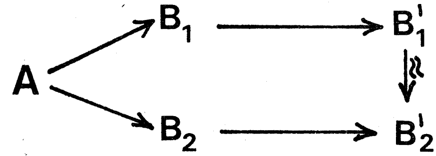


where A', B', C', C'' are iterated ultrapowers of A, B, C respectively and $\tilde{C}', \tilde{C}''$ are iterated ultrapowers of C', C''. Then, obtaining the ultrapowers $\tilde{A}', \tilde{B}'$ from A', B' in the same way as $\tilde{C}', \tilde{C}''$ were obtained from C', C'' we get the isomorphic iterated ultrapowers of A, B which prove their elementary equivalence.

3. DEFINITION 4. A full subcategory $\hat{\mathscr{C}}$ of a category $\mathscr{C}$ is called a model-completion of $\mathscr{C}$ if:

M1) $\hat{\mathscr{C}}$ is model-consistent with $\mathscr{C}$

M'2) for any monomorphism $A \rightarrow B_1$, $A \rightarrow B_2$ with B_1, B_2 in $\hat{\mathscr{C}}$, there exist isomorphic iterated ultrapowers B_1', B_2' of B_1, B_2 such that the following diagram is commutative:



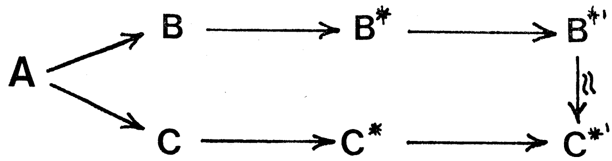
M3) $\hat{\mathscr{C}}$ is a maximal full subcategory of $\mathscr{C}$ with properties M1, M'2

REMARKS. It is obvious that property M'2 implies M2.

The uniqueness of the model-completion of $\mathscr{C}$ results in the same way as that of the model-companion.

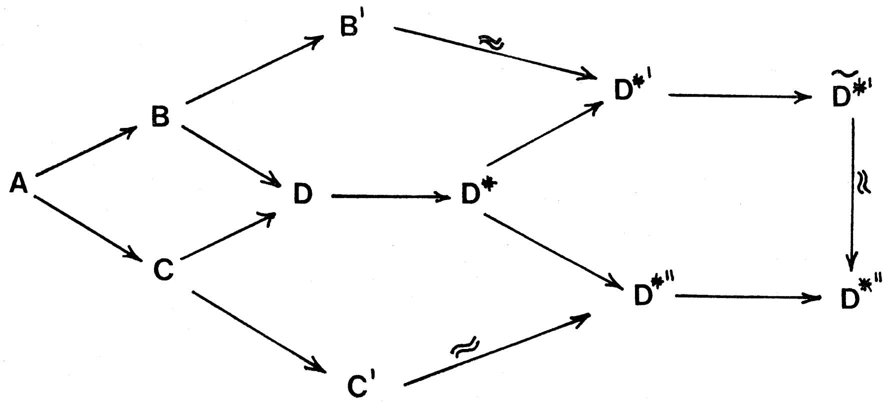
THEOREM 3. *Let $\mathscr{C}^*$ be the model-companion of the category $\mathscr{C}$. Then $\mathscr{C}^*$ is the model-completion of $\mathscr{C}$ if and only if $\mathscr{C}$ has the amalgamation property.*

Proof. Suppose $\mathscr{C}^*$ is the model-completion of $\mathscr{C}$ and let $A \to B, A \to C$ be monomorphisms in $\mathscr{C}$. By M1 and M′2 there results the following commutative diagram:



where B^*, C^* are in $\mathscr{C}^*$ and $B^{*\prime}, C^{*\prime}$ are their iterated ultrapowers. Hence $\mathscr{C}$ has the amalgamation property.

Conversely, let $A \to B, A \to C$ be monomorphisms with B, C in $\mathscr{C}^*$. By the amalgamation property and properties M1, M2, $(\equiv)$ we have the following commutative diagram:



where D^* is in $\mathscr{C}^*$, B', C' and $D^{*\prime}, D^{*\prime\prime}$ are iterated ultrapowers of B, C, D^* respectively, and $\tilde{D}^{*\prime}, \tilde{D}^{*\prime\prime}$ are iterated ultrapowers of $D^{*\prime}, D^{\prime\prime *}$. Then, applying to B', C' the same ultrapower operations that were applied to $D^{*\prime}, D^{*\prime\prime}$ in order to obtain $\tilde{D}^{*\prime}, \tilde{D}^{*\prime\prime}$ we get the desired isomorphic iterated ultrapowers of B, C. Hence $\mathscr{C}^*$ has property M′2.

The maximality property is immediate.

COROLLARY. *Let* $\mathscr{C}, \mathscr{C}', S, T$ *be as in Theorem 1. If* $\hat{\mathscr{C}}'$ *is the model-completion of* $\mathscr{C}'$, *then* $\mathscr{C}$ *has a model-completion containing* $S(\hat{\mathscr{C}}')$.

Proof. From Theorem 1, $\mathscr{C}$ has a model-companion containing $S(\hat{\mathscr{C}}')$. From the adjointness of T, S it results easily that $\mathscr{C}$ has the amalgamation property, so, by Theorem 3, $\mathscr{C}$ has model-completion.

APPLICATIONS

1. Let $\mathscr{B}$ be the category of Boolean algebras and $\mathscr{K}$ the variety generated by an arbitrary crypto-primal algebra [1]. The Boolean power functor

$U : \mathscr{B} \rightarrow \mathscr{K}$ and its left adjoint T constructed in Day [5] have the properties required in Theorem 1, so, using the corollary to Theorem 3 we get some of the results of Bacsich [1].

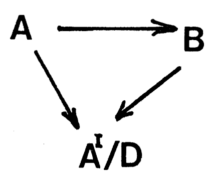
2. Let Luk_n be the category of n-valued Lukasiewicz algebras and $C : \mathrm{Luk}_n \rightarrow \mathscr{B}$ the functor which associates to each algebra L in Luk_n the Boolean algebra of its chrysippian elements [9]. C has a right adjoint D constructed in [10] which fulfills the conditions in Theorem 1. Applying again the Corollary to Theorem 3 we obtain the following: the class of Lukasiewicz algebras has a model-completion which is the subcategory $D(\mathscr{B}_{at})$ of the category of n-valued Post albegras ($\mathscr{B}_{at}$ is the class of atomless Boolean algebras).

5. In a category closed under directed limits and direct products one can also define an analogue to the class of existential structures and obtain results similar to those in Paragraph 1.

DEFINITION 5. Let $\Sigma_{\mathscr{C}}$ be a full subcategory of $\mathscr{C}$. $\Sigma_{\mathscr{C}}$ is called an existential subcategory of $\mathscr{C}$ if:

1) $\Sigma_{\mathscr{C}}$ is model-consistent with $\mathscr{C}$;

2) for every monomorphism $A \rightarrow B$ in $\Sigma_{\mathscr{C}}$, there exist an ultrapower A^I/D and a monomorphism $B \rightarrow A^I/D$ such that the following diagram is commutative:



(the monomorphism $A \rightarrow B$ is existential);

3) $\Sigma_{\mathscr{C}}$ is maximal with properties 1, 2.

PROPOSITION 2. *Let* Σ *be a full subcategory of* $\mathscr{C}$ *with property* 2. *The following statements are equivalent:*

(i) Σ *is an existential subcategory;*

(ii) *any object* A *in* $\mathscr{C}$ *with the property that every monomorphism from* A *to an object in* Σ *is existential, is in* Σ.

Using Proposition 2, the uniqueness of the existential subcategory of $\mathscr{C}$ results in the same way as that of the model-companion.

THE REM 4. *Let* $\mathscr{C}$, $\mathscr{C}'$ *be categories closed under directed limits and direct powers and* $T : \mathscr{C} \rightarrow \mathscr{C}'$, $S : \mathscr{C}' \rightarrow \mathscr{C}$ *a pair of adjoint functors with properties as in Theorem 1. If* $\mathscr{C}'$ *has an existential subcategory* $\Sigma_{\mathscr{C}'}$, *then* $S(\Sigma_{\mathscr{C}'})$ *has properties 1,2 and* $\mathscr{C}$ *has an existential subcategory.*

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