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Lattice Measures, Realcompactness and Pseudocompactness

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Matematica. — *Lattice Measures, Realcompactness and Pseudo-compactness.* Nota II di MARTIN KERNER, presentata (*) dal Corrisp. G. ZAPPA.

RIASSUNTO. — Nella Nota I avevamo introdotto una topologia nello spazio delle misure $\mathscr{L}$–regolari. La base per gli insiemi chiusi in questo spazio è un reticolo e noi mostriamo che questo reticolo è T — 2 se e solo se è normale. Consideriamo poi misure fissate in punti, e mostriamo che sotto certe condizioni esse forniscono un'immagine omomorfa dello spazio. Quindi, estendiamo i nostri risultati a prodotti di reticoli. I principali risultati del lavoro sono teoremi riguardanti la pseudocompattezza e realcompattezza reticolare che generalizzano risultati di Glicksberg a Varadarajan.

In the first part of this paper we discussed properties of lattice regular measures-measures, μ, with the property that $\mu(A) = \sup \mu(L)$, $A \supset L$, $L \in \mathscr{L}$, $\mathscr{L}$ a lattice.

The regularity of our measures ties properties of the measure to properties of the lattice. We now give a measure theoretic characterization of normal lattices.

THEOREM 3.4. *Let $\mathscr{L}$ be a normal lattice. Suppose that μ is a measure on $\mathscr{A}(\mathscr{L})$ and ν and ρ are $\mathscr{L}$-regular measures on $\mathscr{A}(\mathscr{L})$. If $\mu \leq \nu$ on $\mathscr{L}$, and $\mu \leq \rho$ on $\mathscr{L}$, then $\nu = \rho$.*

Proof. Suppose $\nu \neq \rho$. Then there exists $B \in \mathscr{A}(\mathscr{L})$ such that $\nu(B)=0$, $\rho(B) = 0$, and $\rho(B') = 1$. $B \supset D \in \mathscr{L}$ with $\nu(D) = 1$, $B' \supset C \in \mathscr{L}$ with $\rho(C) = 1$. By normality, there exists $H, G \in \mathscr{L}$ with $H' \supset D$, $G' \supset C$ and $H' \cap G' = \varnothing$. Taking complements, $H \cup G = X$ and $\mu(H) = 1$ or $\mu(G) = 1$. If $\mu(H) = 1$, $\nu(H) = 1$, but $\nu(H') = 1$. Contradiction. Similarly, if $\mu(G) = 1$ than $\rho(G) = 1$ but $\rho(G') = 1$. Contradiction.

COROLLARY. *Let $\mathscr{L}$ be a normal lattice. If* F, *a prime $\mathscr{L}$ filter is contained in both* H *and* G, H, G$\mathscr{L}$ *ultrafilters, then* $H = G$.

Proof. By Theorems 2.1 and 2.3, there exists a measure μ and $\mathscr{L}$ regular measures ν and ρ such that $F = \{A \mid \mu(A) = 1\}$, $H = \{B \mid \nu(B) = 1\}$ and $G = \{C \mid \rho(C) = 1\}$. Since $\mu \leq \nu$, $\mu \leq \rho$, $H = G$.

THEOREM 3.5. *If* $(\mu \leq \nu, \mu \leq \rho, \nu, \rho$ *$\mathscr{L}$-regular*$) \Rightarrow \nu = \rho$, *then $\mathscr{L}$ is normal.*

Proof. Assume that $\mathscr{L}$ is not normal. There exist $A, B \in \mathscr{L}$ such that $(x' \in \mathscr{L}' \supset A, y' \in \mathscr{L}' \supset B) \Rightarrow x' \cap y' \neq \varnothing$. Let $G = \{x' \in \mathscr{L}' \mid x' \supseteq A\}$ and let $H = \{y' \in \mathscr{L}' \mid y' \supseteq B\}$. $G \cup H$ is a filter and so $\exists \mathscr{L}$-ultrafilter, $K \supset G \cup H$. Associated with K is an $\mathscr{L}$-regular measure, τ. Let $F = \{y \in \mathscr{L} \mid \tau(y) = 1\}$.

(*) Nella seduta del 13 dicembre 1975.

F is prime and associated with it is a measure, μ. It is straightforward to show that $F \cup \{A\}$ and $F \cup \{B\}$ are filters and they are therefore contained in ultrafilters K_1 and K_2 with associated $\mathscr{L}$-regular measures ν and ρ. Now $\mu \leq \nu$, $\mu \leq \rho$ and $\nu \neq \rho$ since $\nu(A) = 1$, $\rho(A) = 0$. But this is equivalent to the claim of the theorem.

If we let $\{\mathscr{W} = \{W(A) \mid A \in \mathscr{L}\}$ we notice that $\mathscr{W}$ is itself a lattice in $I_R(\mathscr{L})$. The next results use the characterization of normal lattices just discussed to derive properties of this lattice and relate it to properties of $\mathscr{L}$. We need two preliminary lemmas.

LEMMA 3.1. *There exists a 1 : 1 correspondence between* $I_R(\mathscr{L})$ *and* $I_R(\mathscr{W})$.

Proof. The correspondence between $\bar{\mu} \in I_R(\mathscr{W})$ and $\mu \in I_R(\mathscr{L})$ is given by $\bar{\mu}(W(A)) = \mu(A)$, $A \in \mathscr{A}(\mathscr{L})$.

LEMMA 3.2. *Let* $\bar{\mu} \in I(\mathscr{W})$. ($\bar{\mu}$ *is a measure, but not necessarily* $\mathscr{W}$*-regular*). *Than* $\tau \in \bigcap_{\bar{\mu}(W(A))=1} W(A)$ *iff* $\mu \leq \tau$ *on* $\mathscr{L}$, *and* $\tau \in I_R(\mathscr{L})$.

Proof. *a*) If $\tau \in \bigcap_{\bar{\mu}(W(A))=1} W(A)$, than $\tau \in I_R(\mathscr{L})$ and $\mu(A) = 1 \Rightarrow \Rightarrow \bar{\mu}(W(A)) = 1 \Rightarrow \tau \in W(A) \Rightarrow \tau(A) = 1$;

b) If $\mu \leq \tau$ on $\mathscr{L}$, $\tau \in I_R(\mathscr{L})$, then if $\bar{\mu}(W(A)) = 1$ then $\mu(A) = 1 \Rightarrow \Rightarrow \tau(A) = 1 \Rightarrow \tau \in W(A)$.

THEOREM 3.6. $\mathscr{W}$ $T - 2$ *iff* $\mathscr{W}$ *is normal*.

Proof. *a*) Assume $\mathscr{W}$ is normal. If $\mu_1 \neq \mu_2 \in I_R(\mathscr{L})$, $\exists A \in \mathscr{L}$ with $\mu_1(A) = 1$ and $B \in \mathscr{L}$ $A' \supset B$ with $\mu_2(B) = 1$. By the normality of $\mathscr{W}$ there exists $(W(C))' \supset W(A)$, $W(D))' \supset W(B)$, non intersecting, with $\mu_1 \in (W(C))'$, $\mu_2 \in (W(D))$, and $\mathscr{W}$ is $T - 2$.

b) Assume $\mathscr{W}$ is $T - 2$. Suppose $\bar{\mu} < \bar{\tau}_1$, $\bar{\mu} < \bar{\tau}_2$ on $\mathscr{W}$, with $\bar{\mu} \in I(\mathscr{W})$, $\bar{\tau}_1, \bar{\tau}_2 \in I_R(\mathscr{W})$. $\bar{\mu}(W(A)) = 1 \Rightarrow \bar{\tau}_1(W(A)) = \tau_1(A) = \bar{\tau}_2(W(A)) = = \tau_2(A) = 1$. (We are using Lemma 3.1) We now use a characterization of $T - 2$ lattices due to Frolik. A lattice, $\mathscr{L}$, is $T - 2$ iff, for all $0 - 1$ measures, μ, on $\mathscr{A}(\mathscr{L})$, $\cap \{A \subset \mathscr{L} \mid \mu(A) = 1\} = \varnothing$ or a point. $\tau_1, \tau_2 \in \bigcap_{\bar{\mu}(W(A))=1} W(A)$ and by this characterization $\tau_1 = \tau_2$. By Theorem 3.5, $\mathscr{W}$ is normal.

THEOREM 3.7. *Suppose the lattice*, $\mathscr{L}$, *has the following property*: $\forall A \in \in \mathscr{A}(\mathscr{L})\ \exists L \in \mathscr{L} : L \subset A$. *Then* $(I_R(\mathscr{L}), O_W)$ $T - 2 \Rightarrow \mathscr{L}$ *is normal*.

Proof. Suppose $\mu \in I(\mathscr{L})$, $\nu_1, \nu_2 \in I_R(\mathscr{L})$, $\mu \leq \nu_1, \nu_2$ on $\mathscr{L}$. Consider $\bar{\mu}, \bar{\nu}$, and $\bar{\nu}_2$ as in Lemma 3.1. For $W(A)$, $A \in \mathscr{L}$, $\bar{\mu}(W(A)) = 1 \Rightarrow \mu(A) = = 1 \Rightarrow \nu_1(A)) = 1 \Rightarrow \bar{\nu}_1(W(A)) = 1$. So $\bar{\mu} \leq \bar{\nu}_1$ and $\bar{\mu} \leq \bar{\nu}_2$ on $\mathscr{W}$. $(I_R(\mathscr{L}), O_W)$ $T - 2 \Rightarrow \mathscr{W}$ is $T - 2$. By Theorem 3.6. $\mathscr{W}$ normal $\Rightarrow$ by Theorem 3.4 $\bar{\nu}_1 = \bar{\nu}_2 \Rightarrow \nu_1 = \nu_2$, and $\mathscr{L}$ is normal by Theorem 3.5.

Lastly, we consider measures fixed at points, i.e. measures which evaluate to 1 if a set includes a certain ponit, and to zero if a set excludes it. The next results show that when a lattice is related to the topology of a space, measures fixed at points yield a homeomorphic image of the space. Brooks [4] has considered similar questions using filters. Our approach is measure theoretic and, using the measure-filter correspondence, the results on filters are obtained as corollaries. On our discussion, μ_x will be the measure fixed at the point x. $\mathscr{L}$ is assumed atom disjunctive.

LEMMA 3.3. *If $\mathscr{L}$ is atom disjunctive, than* $U_x = \{L \in \mathscr{L} \mid x \in L\}$ *is an ultrafilter.*

Proof. If U_x is not an ultrafilter, $\exists G \supset U_x$, $L \in G$, $L \notin U_x \Rightarrow x \notin L$. By disjunctive property, $\exists L_1 \in \mathscr{L}$, $x \in L_1$, $L \cap L_1 = \varnothing$. But $G \supset U_x \Rightarrow L_1 \in G$. $L, L_1 \in G \Rightarrow L \cap L_1 \in G \Rightarrow \varnothing \in G$. Contradiction.

COROLLARY. *If $\mathscr{L}$ is atom disjunctive, μ_x is $\mathscr{L}$-regular.*

Proof. μ_x is the measure associated with the ultrafilter U_x.

THEOREM 3.8. *$\mathscr{L}$ is* $T-1$ *iff* $\varphi : X \to I_R(\mathscr{L})$ *by* $\varphi(x) = \mu_x$ *is* $1:1$.

Proof. Assume $x_1 \neq x_2$. By T—1 property $\exists A$ with $x_1 \in A$, $x_2 \notin A$. $\mu_{x_1} \neq \mu_{x_2}$ since they take different values on A. Conversely, if φ is $1:1$, $x_1 \neq x_2 \Rightarrow \exists A$ such that $\mu_{x_1}(A) \neq \mu_{x_2}(A) \Rightarrow$ one of the points is in A, the other is not $\Rightarrow \mathscr{L}$ is $T-1$.

THEOREM 3.9. $\overline{\{\mu_x\}}$ *in the* O_W *topology* $= I_R(\mathscr{L})$ *i.e. measures fixed at a point are dense in the space of* $0-1$ *$\mathscr{L}$-regular measures.*

Proof. For $\mu \in I_R(\mathscr{L})$, a basic open set containing $\mu = (W(A))'$. Choose $x \in A$. By the atom disjunctive property $\exists L_1 \in \mathscr{L}$ such that $L_1 \cap A = \varnothing$. This implies that $\mu_x \in (W(A))'$. Since μ was arbitrary, $\overline{\{\mu_x\}} = I_R(\mathscr{L})$.

THEOREM 3.10. $\varphi : x \to \mu_x$ *is continuous iff each* $L \in \mathscr{L}$ *is closed in* X.

Proof. A basic open set containing $\mu_x = (W(A))' \cap \varphi(X)$. φ^{-1} of this set is A', $A \in \mathscr{L}$, which is open in X by assumption. Conversely, assume φ is continuous. Then $\varphi^{-1}(W(A) \cap \varphi(X))$ is closed in X. But this is just A.

THEOREM 3.11. *φ is a homeomorphism*: $X \to I_R(\mathscr{L})$ *iff $\mathscr{L}$ is a* $T-1$ *base lattice (A base for the closed sets in* X).

Proof. φ is $1:1$ and continuous by 3.8 and 3.10. Choose F, closed in X. $F = \cap L_i$, $L_i \in \mathscr{L}$. $\varphi(F) = \cap \varphi(L_i) = \cap (W(L_i) \cap \varphi(X))$. (This is obtained by applying φ to the formula $\varphi^{-1}((W(A))' \cap \varphi(X)) = A'$ which we used in the proof of Theorem 3.10). This set is closed, so φ is bijective, continuous and closed which is equivalent to homeomorphic.

Conversely, if φ is a homeomorphism, than $\mathscr{L}$ is $T-1$ by 3.8, above. Choose F, closed in X. $\varphi(F)$ is closed. $\varphi(F) = \cap (W(A_i) \cap \varphi(X))$, $A_i \in \mathscr{L}$.

This implies that $F = \cap A_i$, and $\mathscr{L}$ is a base for the closed sets of X.

4. Products

We have already discussed, in section 2, how a multiplicative system may be extended to a lattice. An intuitive example of a multiplicative system is $\mathscr{L}_1 \times \mathscr{L}_2$. In this section we show how to extend measures on $\mathscr{L}_1$ and $\mathscr{L}_2$ to a measure on the product space. For a filter approach to similar questions, see Kost [9]. Our approach is measure theoretic and, using the measure-filter correspondence, we get Kost's results as corollaries. Given $X_1, \mathscr{L}_1$ and μ, defined on $\mathscr{A}(\mathscr{L}_1)$, and $X_2, \mathscr{L}_2$ and ν, defined on $\mathscr{A}(\mathscr{L}_2)$, let $\mathscr{M} = \{A \times B \subset X_1 \times X_2, A \in \mathscr{L}_1, B \in \mathscr{L}_2\}$. $\mathscr{M}$ is a multiplicative system. Let $\mathscr{L}(\mathscr{M}) = \{\overset{n}{\underset{1}{\cup}} A_i \times B_i A_i \in \mathscr{L}_1, B_i \in \mathscr{L}_2\}$. For $K = A \times B$, define $\rho(K) = \mu(A)\nu(B)$. Define $\rho(\overset{n}{\underset{1}{\cup}} A_i \times B_i) = \sum_1^n \mu(A_i)\nu(B_i)$, $A_i \cap A_j = \varnothing$, $B_i \cap B_j = \varnothing$ for $i \neq j$. We note that any union in $\mathscr{L}(\mathscr{M})$ may be written as a disjoint union. We now have defined a measure on $\mathscr{L}(\mathscr{M})$, and by the discussion preceding Theorem 2.3 we can extend this measure to a measure, ρ, defined on $\mathscr{A}(\mathscr{L}(\mathscr{M}))$.

THEOREM 4.1. *For every pair of* 0 — 1 *measures,* μ *on* $\mathscr{A}(\mathscr{L}_1)$ *and* ν *on* $\mathscr{A}(\mathscr{L}_2)$, *there corresponds a measure,* ρ, *on* $\mathscr{A}(\mathscr{L}(\mathscr{M}))$ *and conversely.*

Proof. We have already described the correspondence in one direction. Assume that we are given ρ on $\mathscr{A}(\mathscr{L}(\mathscr{M}))$. For $A \in \mathscr{A}(\mathscr{L}_1)$, define $\mu(A) = \rho(A \times X_2)$. If $A_i \cap A_j = \varnothing$, $i \neq j$, $\mu(\overset{n}{\underset{1}{\cup}} A_i) = \rho((\cup A_i) \times X_2) = \Sigma_i (A_i \times X_2) = \Sigma \mu(A_i) \cdot \mu$ is therefore a measure on $\mathscr{A}(\mathscr{L}_1)$.

In an entirely similar manner, we can define $\nu(B) = \rho(X_1 \times B)$, $B \in \mathscr{A}(\mathscr{L}_2)$.

Having topologized $I_R(\mathscr{L}_1)$, $I_R(\mathscr{L}_2)$ and $I_R(\mathscr{L}(\mathscr{M}))$, we can refine our result.

THEOREM 4.2. *Let* $f: I_R(\mathscr{L}_1) \times I_R(\mathscr{L}_2) \to I_R(\mathscr{L}(\mathscr{M}))$ *by* $f(\mu, \nu) = \rho$ *where* ρ *is defined as above. Then* f *is a homeomorphism.*

Proof. A basic closed set in $I_R(\mathscr{L}(\mathscr{M}))$ is $W(\overset{n}{\underset{1}{\cup}} A_i \times B_i) = \overset{n}{\underset{1}{\cup}} W(A_i \times B_i) \cdot f^{-1}(\overset{n}{\underset{1}{\cup}} W(A_i \times B_i) = \overset{n}{\underset{1}{\cup}} (f^{-1}(W(A_i \times B_i))$. Now $W(A \times B) = \{\rho \mid \rho(A \times B) = 1\} = \{\mu \mid \mu(A) = 1\} \times \{\nu \mid \nu(B) = 1\} = W(A) \times W(B) \cdot f^{-1}(W(A_i \times B_i)$ is therefore $W(A_i) \times W(B_i)$, a closed set in $I_R(\mathscr{L}_1) \times I_R(\mathscr{L}_2)$ and f is continuous.

Since $f^{-1}(W(A \times B)) = W(A) \times W(B)$, apply f to both sides to get $f(W(A) \times W(B)) = W(A \times B) f$, therefore, takes basic closed sets into closed sets and is a closed map. f is now closed, continuous and bijective, equivalent to being a homeomorphism.

5. LATTICE PSEDOCOMPACTNESS

In this chapter we generalize certain results of Varadarajan and Glicksberg. All lattices in this section will be normal, atom disjuctive δ-lattices.

DEFINITION. A real valued function is lattice-continuous if the inverse image of a closed set in R is in the lattice.

LEMMA 5.1. *If* $L_1 \cap L_2 = \varnothing$, $L_1, L_2 \in \mathscr{L}$, $\exists$: *a bounded, $\mathscr{L}$-continuous* f, $0 \leq f \leq 1$, $L_1 = f^{-1}(0)$, $L_2 = f^{-1}(1)$.

Proof. This is Urysohn's lemma applied to normal lattices. For a complete proof, see [1], p. 317 ff.

DEFINITION. A sequence $\{L_i\}$ is $\mathscr{L}$-regular if *a)* $L_i \uparrow X$ *b)* For all n, there exists $U_n = L_{k'}$, $L_k \in \mathscr{L}$, such that $L_n \subset U_n \subset L_{n+1}$. A regular sequence is called terminating if $L_n = X$ for some n.

THEOREM 5.1. *Let* $\{L_i\}$ *be an $\mathscr{L}$-regular sequence. Let* $\tau(\mathscr{L}) = \{\cap L_k \mid L_k \in \mathscr{L}\}$. *Then* $C \in \tau(\mathscr{L})$ *iff* $C \cap L_i \in \tau(\mathscr{L})$ *for all* L_i *in the $\mathscr{L}$-regular sequence. As a result, f is $\tau(\mathscr{L})$ continuous iff it is $\tau(\mathscr{L})$ continuous on each L_i in the $\mathscr{L}$-regular sequence.*

Proof. In one direction the result is clear. Assume $C \cap L_i \in \tau(\mathscr{L})$. If $L_n \subset U_n \subset L_{n+1}$, choose $x \in X - C$, $x \in U_n$ for some n. $x \notin C \cap L_{n+1} \in \tau(\mathscr{L})$. Let $C \cap L_{n+1} = \cap L_k$. $x \in \cap (L_k)' = \cup L_{k'} \Rightarrow \exists L_{j'} \in \{L_{k'}\}$ with $x \in L_{j'}$. Let $G_1 = L_{j'} \cap U_n = \overline{L}' \in \mathscr{L}'$. It is easy to show that $G_1 \cap C = \varnothing$. For each $x \in X - C$, generate $\overline{G}$ with $x \in \overline{G}$, $\overline{G} \cap C = \varnothing$

$$X - C = \cup \overline{G} \quad , \quad C = (\cup \overline{G})' = \cap \overline{G}' = \cap L_i \in \tau(\mathscr{L}).$$

THEOREM 5.2. *Let* $\{L_n\}$ *be an $\mathscr{L}$-regular sequence, and let* $\{t_n\}$ *be an increasing sequence of numbers. Then* $\exists g$: *a* $\tau(\mathscr{L})$*-continuous function, defined on* X, *such that, for each* n, $L_n = \{x \mid g(x) \leq t_n\}$.

Proof. We define f_i by induction satisfying: 1) $f_1 = t_1$ on L_1; 2) $f_n = t_1$ on L_1, t_n on $L_n - U_{n-1}$ and on $U_i - L_i$, $t_i < f_n < t_{i+1}$; 3) f_i is $\mathscr{L}$-continuous on L_i; 4) f_n extends $f_{n-1} \cdot f_1$ is easily constructed. Assume that $f_1 \cdots f_n$ satisfy 1)-4).

Then $\bar{g}$, continuous on L_{n+1}, $= t_n$ on L_n and t_{n+1} on $L_{n+1} - U_n$ exists by an application of Lemma 5.1. Let $f_{n+1} = f_n$ on L_n and $\bar{g}$ on $L_{n+1} - L_n$. Define g on X by $g = f_n$ on L_n. It is clearly $\tau(\mathscr{L})$ continuous on L_n, and by Theorem 5.1 it is $\tau(\mathscr{L})$ continuous on X. By construction, $L_n = \{x : g(x) \leq t_n\}$.

THEOREM 5.3. *If* $\{L_n\}$ *is a sequence* $\downarrow \varnothing$, *there exists an $\mathscr{L}$-regular sequence,* $\{L_n^*\}$ *such that* $L_n^* \subset L_n'$ *for each* n.

Proof. Let f_n be bounded, $\mathscr{L}$-continuous, $0 \leq f_n \leq 1$ and $L_n = f_n^{-1}(0)$.

Define $f_n = \max(f_1, \cdots, f_n)$. Let $L_n^* = \{x : f_n^*(x) \geq 1/n\}$ and $U_n = \{x : f_n(x) > 1/(n+1)\}$. It is straightforward to show that $\{L_n^*\}$ is the desired sequence.

DEFINITION. Given a space, X, and an associated lattice, $\mathscr{L}$, X is $\mathscr{L}$-pseudocompact if every real valued, $\mathscr{L}$-continuous function is bounded. The next result is the major one in this section. It generalizes a Theorem of Glicksberg that is cited by Varadarajan and relates pseudocompactness to properties of lattice regular sequences.

THEOREM 5.4. *If* X *is* $\tau(\mathscr{L})$ *pseudocompact, then the following equivalent conditions hold*: I) *Each* $\mathscr{L}$*-regular sequence in* X *terminates.* II) *Each countable covering of* X *by* U *sets has a finite subcovering.* III) *For* X, *Dini's theorem holds. All the above imply the following.* IV) X *is* $\mathscr{L}$*-Pseudocompact.*

Proof. $\tau(\mathscr{L})$ *pseudocompact* $\rightarrow$ *I*). Let $\{L_i\}$ be $\mathscr{L}$ regular. By Theorem 5.2 $\exists$: $\tau(\mathscr{L})$ continuous g with $L_n = \{x : g(x) \leq t_n\}$. Since X is $\tau(\mathscr{L})$ pseudocompact, g is bounded. If g is bounded, $\{L_i\}$ must terminate. $I \leftrightarrow II$) II may be restated: The intersection of any decreasing sequence of sets is not empty. By Theorem 5.3 this is equivalent to every $\mathscr{L}$-regular sequence terminating. $I \leftrightarrow III$) Let $\{f_n\}$ be bounded, $\mathscr{L}$ continuous, $\downarrow 0$. If $L_n = \{x : f_n(x) \geq \varepsilon\}$, $\{L_n\} \downarrow \varnothing$. Since $I \rightarrow II$, $L_n = \varnothing$ for $n \geq n_0 \rightarrow f_n < \varepsilon$ for $n \geq n_0$. Thus $f_n \downarrow 0$ uniformly and III holds. Conversely, if $\{L_n\}$ is non-terminating and regular, $\exists \{f_n\}$, f_n bounded, $\mathscr{L}$ continuous with $0 \leq f_n \leq 1$ and $f_n \downarrow 0$ such that $L'_{n+1} \subset f_n^{-1}(1)$. We can choose $x_n \in L'_{n+1}$ for each n. Since $f_n(x_n) = 1$, f cannot converge uniformly and III $\rightarrow$ I. $I \rightarrow IV$) Let g be bounded, $\mathscr{L}$ continuous. $L_i = \{x : g(x) \leq i, i = 1, 2 \cdots\}$ is an $\mathscr{L}$ regular sequence. I implies that such a sequence terminates which implies that g is bounded and IV is true.

COROLLARY. X *is* $\tau(\mathscr{L})$ *pseudocompact iff each* $\tau(\mathscr{L})$ *regular sequence terminates.*

Proof. We note that $\tau(\tau(\mathscr{L})) = \tau(\mathscr{L})$. If the lattice in Theorem 5.4 is replaced by $\tau(\mathscr{L})$ we get: $\tau(\mathscr{L})$ pseudocompact $\rightarrow \tau(\mathscr{L})$ regular sequences terminate $\rightarrow$ IV) X is $\tau(\mathscr{L})$ pseudocompact.

6. LATTICE REALCOMPACTNESS

The major result of this section is to generalize a theorem of Varadarajan, which we obtain as a corollary.

DEFINITION. X is $\mathscr{L}$ realcompact iff every $\mathscr{L}$-regular, σ-smooth $0-1$ measure is fixed at a point.

We note that the measure may be defined on either $\mathscr{A}(\mathscr{L})$ or $\sigma(\mathscr{L})$. It can be shown [8] that every σ smooth $\mathscr{L}$ regular $0-1$ measure on $\mathscr{A}(\mathscr{L})$ is fixed at a point iff such measures defined on $\sigma(\mathscr{L})$ are also fixed at a point.

Our result concerns lattice realcompact properties of subsets. Our setting for these results is the following: A space X, a lattice $\mathscr{L}$ on X, a subset $A \subset X$ and a lattice $\mathscr{L}_A$ on A. $\mathscr{L}$ is a normal base lattice for a topology on X, and $\mathscr{L}_A$ is a normal base lattice for a topology on A. We assume that $\tau(\mathscr{L}_A) = \tau(\mathscr{L}) \cap A$. This is equivalent to the topology on A being the same as the topology A inherits from X.

NOTATION. *$\rho(\mathscr{L})$ is the smallest collection of sets, containing $\mathscr{L}$ which is closed under countable intersection and union.*

THEOREM 6.1. *Assume the conditions on $\mathscr{L}$, $\mathscr{L}_A$ discussed above. Let X be $\mathscr{L}$-realcompact. Suppose that $\rho(\mathscr{L}) = \sigma(\mathscr{L})$. If $A \subset X$, a sufficient condition for A to be $\mathscr{L}_A$ realcompact is:* $\forall x \in X - A \ \exists : E \in \sigma(\mathscr{L}) \ni : E \supset A$ *and* $x \in X - E$.

THEOREM 6.2. *Assume the conditions on $\mathscr{L}$, $\mathscr{L}_A$ discussed above. If A has the property that $A \cap L \in \mathscr{L}$, whenever $L \in \mathscr{L}$ than, in order for $A \subset X$ to be realcompact, a sufficient condition is:* $\forall x \in X - A \exists : E \in \sigma(\mathscr{L}) \ni : E \supset A$ *and* $x \in X - E$.

Note. The different conditions in the theorems are each sufficient to guarantee that a certain measure is regular. The following proof establishes both theorems.

NOTATION. *$I_R^\sigma(\mathscr{L})$ is the set of σ-smooth, $\mathscr{L}$ regular, $0-1$ measures.*

Proof. Let $m_0 \in I_R^\sigma(\mathscr{L} \cap A)$. For $K \in \sigma(\mathscr{L})$, define m by $m(K) = m_0(K \cap A)$. Since $\sigma(\mathscr{L} \cap A) = \sigma(\mathscr{L}) \cap A$, our measure is defined on the appropriate set.

We claim that $m \in I_R^\sigma(\mathscr{L})$. It is easy to show that m is $\mathscr{L}$-smooth. To prove the $\mathscr{L}$-regularity we consider two cases: *a)* $A \cap L \in \mathscr{L}$, whenever $L \in \mathscr{L}$.

Proof. $m(K) = 1 \Rightarrow m_0(K \cap A) = 1 \Rightarrow \exists : \bar{L} \cap A \subset K \cap A$ and $m_0(\bar{L} \cap A) = 1$. If $\bar{L} \cap A \in \mathscr{L}$, then, since $K \supset \bar{L} \cap A$, $m(K) = \sup_{K \supset L, L \in \mathscr{L}} m(L)$;

case *b)* If A is arbitrary but $\sigma(\mathscr{L}) = \rho(\mathscr{L})$.

Consider K = the collection of subsets of $\sigma(\mathscr{L})$ on which m is $\mathscr{L}$ regular. It is not difficult to show that K contains $\mathscr{L}$ and is closed under countable unions and countable intersections. This collection therefore $\supset \rho(\mathscr{L}) = \sigma(\mathscr{L})$.

Since $K \subset \sigma(\mathscr{L})$, m is $\mathscr{L}$-regular on $\sigma(\mathscr{L})$.

Since X is realcompact, m is fixed at a point $x_0 \in X$.

Claim: $x_0 \in A$. proof: if $x_0 \in X - A$ than $\exists : E \in \sigma(\mathscr{L})$ with $x_0 \in X - E$. Than $m(X - E) = m_0((X - E) \cap A) = m_0(\varnothing) = 0$. Contradiction.

The definition of m_0 guarantees that m_0 is also degenerate at x_0 which shows that A is $\mathscr{L} \cap A$ realcompact. *Claim*: A is $\mathscr{L}_A$ realcompact.

Proof. We show that $\bar{m}(K) = 1$ iff $x_0 \in K$, for $K \in \sigma(\mathscr{L}_A)$. It is sufficient to show that $\bar{m}(U_0) = 1$ if $x_0 \in U_0$ and $U_0 = L'_A$, $L_A \in \mathscr{L}_A$.

Let $x_0 \in U_0 \cdot U_0$ is open in A which implies that $U_0 = G \cap A$, where G is open in X. (Here we are using the restriction on the topologies). Since

$x_0 \in G$, $U = L'$, $L \in \mathscr{L}$, $\ni : x_0 \in U \subset G$. x_0 is then in $U \cap A \subset U_0$. Since $U \cap A \in \sigma (A \cap \mathscr{L})$, and $x_0 \in U \cap A$, $\overline{m}(U \cap A) = 1$. ($\overline{m}$, restricted to $\sigma(\mathscr{L} \cap A)$) must degenerate at x_0 by the above discussion. Consequently, $\overline{m}(U_0) = 1$, and the theorem is proved.

Given a space X, assumed completely regular, Hausdorff, let $Z(X) = \{x : f(x) = 0\}$ for some continuous, real valued f. We note that $Z(X)$ is a lattice, and $\sigma(Z(X))$ = Baire sets of the space.

COROLLARY (Varadarajan). *Let* X *be* Z (X)-*realcompact.* (*Varadarajan calls such a space a* Q *space*). *In order that* X_0 *be* Z (X) *realcompact, it is sufficient that for any point* $x \in X - X_0$, *there exists a Baire subset of* X, *say* E, *such that* $X_0 \subset E$ *and* $x \in X - E$.

Proof. $\sigma(Z(X)) = \rho(Z(X))$. All the conditions of Theorem 6.1 now hold.

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