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Strong oscillation of elliptic systems of second order partial differential equations

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RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

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Presiede il Presidente della Classe Beniamino Segre

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *Strong oscillation of elliptic systems of second order partial differential equations.* Nota di Samuel M. Rankin, III, presentata (*) dal Socio C. Miranda.

Riassunto. — Si dà una condizione sufficiente ad assicurare il carattere fortemente oscillatorio dei sistemi ellittici del tipo

$$\sum_{i,j=1}^{n} D_i (A_{ij}(x) D_j u) + C(x) u = 0 .$$

Con questo risultato si estendono vari criteri noti relativi al caso di una sola equazione.

Oscillation theory for elliptic partial differential equations and elliptic systems of partial differential equations of second order has developed rapidly during recent years with the papers of Clark and Swanson [1], Headly [3], Headly and Swanson [4], Kreith [5, 6, 7], Kreith and Travis [8, 9], Swanson [11, 12], and Travis [13, 14]. The papers are separated into three basic types: (*a*) those concerned with Sturm-Picone comparison theorems for scalar equations ([1]), [5], [6]); (*b*) those concerned with sufficient conditions for non-oscillation and oscillation of scalar equations ([9], [12], [13]); (*c*) those concerned with elliptic systems of type (3) below ([7], [8], [11], [14]).

The main result of this paper gives a sufficient condition for strong oscillation of the equation

$$\text{(1)} \qquad Lu = \sum_{i,j=1}^{n} D_i (A_{ij}(x) D_j u) + C(x) u = 0$$

(*) Nella seduta del 15 novembre 1975.

where $A_{ij}(x)$ $(i, j = 1, \cdots, m)$ and $C(x)$ are $m \times m$ matrix functions defined in E^n. The solution $u(x) = (u_1(x), \cdots, u_m(x))^T$ is an $m \times 1$ vector function. To be explicit, we show that nodal oscillation of an equation of type (2)

$$lu = \sum_{i,j=1}^{n} D_i(a_{ij}(x) D_j v) + c(x) v = 0 \tag{2}$$

implies the nodal oscillation of equation (1). Here $a_{ij}(x)$ $(i, j = 1, \cdots, m)$ and $c(x)$ are real valued scalar functions.

Comparison theorems between two equations of type (1) and between equations of type (1) and equations of the form

$$\bar{L}V = \sum_{i,j=1} D_i(A_{ij}(x) D_j V) + C(x) V = 0 \tag{3}$$

where V is an $m \times m$ matrix function are also developed in this paper.

Our techniques will be based on the methods of the calculus of variations in characterizing the eigenvalues and eigenfunctions of the system

$$\begin{aligned} \sum_{i,j=1}^{n} D_i(A_{ij}(x) D_j z(x)) + C(x) z(x) + \lambda z(x) &= 0 \qquad \text{on } G \\ z(x) &= 0 \qquad \text{on } F(G) \end{aligned} \tag{4}$$

where G is a bounded domain in E^n and F (G) is its boundary. We will use properties of system (4) which are developed in [10].

The following assumptions will be made throughout the paper.

(i) $C(x)$ and each $A_{ij}(x)$ are m-square real matrix functions defined in E^n; the solutions $u(x)$ of equation (1) are $m \times 1$ vectors.

(ii) $A_{ij} = A_{ji}$ and each A_{ij} is symmetric as class $C'(E^n)$.

(iii) The mn-square matrix $(A_{ij}(x))$ is positive definite.

(iv) C is symmetric and continuous.

(v) The notations a_{ij}^{hl} and c^{hl} will be used to denote the hl-th elements of the matricies $A_{ij}(x)$ and $C(x)$, respectively.

A bounded domain $G \subset E^n$ is said to be a nodal domain of a solution u of (1) (resp. (2)), if and only if $u = 0$ on F (G). The equation (1) (resp. (2)) will be called nodally or strongly oscillatory in E^n if for every $R > 0$ there is a a domain G in

$$E_R \stackrel{\text{def}}{=} \{x \in E^n \mid |x| = \sqrt{x_1^2 + \cdots + x_n^2} > R\},$$

such that G is a nodal domain for a solution of (1) (resp. (2)).

The following three theorems are stated without proofs. Their proofs can be found in [10].

THEOREM A. *The linear operator*

$$Lu = \sum_{i,j=1}^{n} D_i (A_{ij}(x) D_j u) + C(x) u$$

has a discrete spectrum in the bounded region $G \subset E^n$, *the smallest eigenvalue of the operator being given by the formula*

$$\lambda_1 = \min_{u \in \Phi} \frac{\int_G \left(\sum_{i,j=1}^{n} D_i u^T A_{ij}(x) D_j u - u^T C(x) u \right) dG}{\int_G \sum_{i=1}^{m} u_k^2(x)\, dG}.$$

The class Φ *is an appropriate set of " admissible functions."*

THEOREM B. *For* $r_1 > 0$ *define* $G(r_1 r) = \{x \mid r_1 < |x| < r\}$, *then the smallest eigenvalue* $\lambda(r)$ *of the system*

$$(5) \qquad \sum_{i,j=1}^{n} D_i (A_{ij}(x) D_j u) + C(x) u = 0 \quad \text{on } G(r_1, r)$$

$$u = 0 \quad \text{on } F(G(r_1, r))$$

satisfies $\lim_{r \to r_1} \lambda(r) = \infty$.

THEOREM C. *Define* $G(r_1, r)$ *as in Theorem B, then the smallest eigenvalue* $\lambda(r)$ *of the system* (5) *depends continuously on* r.

We now state and prove a useful lemma.

LEMMA. *If there exists a solution* $v(x)$ *of equation* (2) *that has a nodal domain* $G \subset E_R$ *for some* $R > 0$, *then there exists a region* $G(a, b)$ *such that* $G(a, b)$ *is a nodal domain for a solution* $z(x)$ *of equation* (2).

Proof. Since $v(x)$ satisfies equation (2) for the region G and $v(x) = 0$ on F(G), zero is an eigenvalue of equation (2) and $v(x)$ is its corresponding eigenfunction. Enclose G in a region of the form $G(a, r) = \{x \mid 0 < a < |x| < r\}$. By classical variational principles [2] the smallest eigenvalue of the problem

$$lu + \lambda u = 0 \quad \text{on } G(a, r)$$

$$u = 0 \quad \text{on } F(G(a, r))$$

is less than or equal to zero. Appealing to Theorems B and C, there exists a $b > a$ such that zero is the smallest eigenvalue of the system

$$lu + \lambda u = 0 \quad \text{on } G(a, b)$$

$$u = 0 \quad \text{on } F(G(a, b)).$$

Thus the proof is complete.

Our main theorem can now be stated.

THEOREM 1. *If the scalar elliptic equation*

$$\sum_{i,j=1}^{n} D_i \left(a_{ij}^{kk}(x) D_j v(x)\right) + c^{kk}(x) v = 0 , \tag{6}$$

is nodally oscillatory for some $k = 1, \cdots, m$ *then equation* (1) *is nodally oscillatory.*

Proof. Since (6) is nodally oscillatory we have by the lemma that there exists for each $R > 0$ a domain $G = G(r_1, r)$ where $r_1 > R$ and such that G is a nodal domain for a solution $v(x)$ of (6). Define $u(x) = v(x)\beta$ where β is a constant $m \times 1$ vector with one in the k-th component and zeros elsewhere, then $u(x) = 0$ on $F(G(r_1, r))$. Now consider the eigenvalue problem (4). If Φ denotes the class of "admissible functions", Theorem A implies that

$$\lambda_1 = \min_{z \in \Phi} \frac{\int_G \left(\sum_{i,j=1}^{n} D_i z^T A_{ij}(x) z - z^T C(x) z\right) dG}{\int_G \left(\sum_{k=1}^{m} z_k^2(x)\right) dG}$$

$$\leq \frac{\int_G \left(\sum_{i,j=1}^{n} D_i u^T A_{ij}(x) D_j u - u^T C(x) u\right) dG}{\int_G \left(\sum_{k=1}^{m} u_k^2(x)\right) dG}$$

$$= \frac{\int_G \left(\sum_{i,j=1}^{n} a_{ij}^{kk} D_i v(x) D_j v(x) - C^{kk} v^2(x)\right) dG}{\int_G v^2(x)\, dG} .$$

Since $v(x)$ satisfies (6) and $v(x) = 0$ on $F(G)$, the last ratio is zero and $\lambda_1 \leq 0$. Now by Theorems B and C, there is a domain $G(r_1, r') \subseteq G$ for which the eigenvalue problem

$$Lu + \lambda' u = 0 \qquad \text{on} \quad G(r_1, r')$$

$$u = 0 \qquad \text{on} \quad F(G(r_1, r'))$$

satisfies $\lambda_1' = 0$. This completes the proof.

From the proof of Theorem 1 we see that the following result can be stated.

THEOREM 2. *If for each* $R > 0$ *there exists a domain* $G \subset E_R$, *a* $k \in \{1, \cdots, m\}$ *and a solution* v *of equation* (6) *such that* $v = 0$ *on* $F(G)$, *then equation* (2) *is nodally oscillatory.*

COROLLARY 1. *If for each* $R > 0$ *there exists a bounded region* $G \subset E_R$ *with piecewise smooth boundary, a continuously differentiable function* z *with* $z(x) = 0$ *on* $F(G)$ *and an integer* $k \in \{1, \cdots, m\}$ *such that*

$$\int_G \left(\sum_{i,j=1}^{n} a_{ij}^{kk} D_i z D_j z - c^{kk} z \right) dG \leq 0, \tag{7}$$

the equation (1) *is strongly oscillatory.*

Proof. Condition (7) implies that the smallest eigenvalue for the system

$$\begin{aligned} \sum_{i,j=1}^{n} D_i (a_{ij}^{kk} D_j u) + c^{kk} u + \lambda u &= 0 \qquad \text{on} \quad G \\ u &= 0 \qquad \text{on} \quad F(G) \end{aligned} \tag{8}$$

is less than or equal to zero. Now by classical variational principles found in [2], there exists a domain $G' \subset G$ such that the smallest eigenvalue λ' for the system (8) defined on G' satisfies $\lambda' = 0$ and the corresponding eigenfunction u' satisfies $u'(x) = 0$ on $F(G)$. Now apply Theorem 2.

We can extend a result of Kreith and Travis [9] to vector equations. Define

$$\beta_k(r) = \int_\Delta r^T(\theta) A_k(r, \theta), \quad r(\theta) \, d\theta$$

$$\alpha_k(r) = \int_\Delta c_k(r, \theta) \, d\theta$$

where Δ denotes the unit $(n-1)$ sphere in E^n the column vector $r(\theta)$ is the exterior unit normal to the sphere Δ at (r, θ) and where $A_k(r, \theta)$ and $c_k(r, \theta)$ denote the matrices $(a_{ij}^{kk}(x))$ and $c^{kk}(x)$, respectively; x is written in terms of hyperspherical coordinates for E^n.

THEOREM 3. *If the ordinary differential equation*

$$\frac{d}{dr}\left(r^{n-1} \beta_k(r) \frac{dv}{dr}\right) + r^{n-1} \alpha_k(r) v = 0$$

for some $k = 1, \cdots, m$ *is oscillatory at* $r = \infty$, *then equation* (1) *is nodally oscillatory at* $|x| = \infty$.

Proof. Equation (6) is nodally oscillatory by the theorem of Travis [13]. Now by Theorem 1 we have the strong oscillation of (1) in E^n.

THEOREM 4. *If for some* $k = 1, \cdots, m$ *equation* (*) $\Delta u + c^{kk} u = 0$ *is nodally oscillatory for all positive* λ *and if* $(a_{ij}^{kk}(x))$ *is bounded as a form in* E^n, *then equation* (1) *is strongly oscillatory.*

Proof. Let a_0 be a positive upper bound for $(a_{ij}^{kk}(x))$. Since (*) is nodally oscillatory for all $\lambda > 0$, there exists for each $R > 0$ a domain $G \subset E_R$ and a solution u of (*) with $\lambda = \frac{1}{a_0}$ such that $u = 0$ on $F(G)$. Now we have

$$\int_G \left(\sum_{i,j=1}^n a_{ij}^{kk} D_i u D_j u - c^{kk} u^2 \right) dG$$

$$\leq \int_G \left(a_0 \sum_{i=1}^n (D_i u)^2 - c^{kk} u^2 \right) dG$$

$$= a_0 \int_G \left(\sum_{i=1}^n (D_i u)^2 - \lambda c^{kk} u^2 \right) dG = 0.$$

By corollary (1), the conclusion follows.

Let $\alpha_k(r) = \frac{1}{\Sigma_n} \int_\Delta c^{kk}(r, \theta) \, d\theta$ where Δ again denotes the full range of angular coordinates and $\Sigma_n = \int_\Delta d\theta$. We then have the following corollary of Theorem 4, which extends another result of Kreith and Travis [9] to vector equations.

COROLLARY 2. *If* $\limsup_{r \to \infty} r \int_r \alpha_k(r) \, dr = \infty$ *and* $(a_{ij}^{kk}(x))$ *is bounded as a form on* E^n *for some* $k = 1, \cdots, m$, *then equation* (1) *is strongly oscillatory.*

Proof. The proof follows from Theorem 4.4 of [9] and Theorem 4.

A result similar to one found in a paper by Swanson [12] for scalar equations can be obtained for vector equations.

THEOREM 5. *Equation* (1) *is strongly oscillatory if for every* $r > 0$ *there exists*

(1) *a bounded region* $M \subset E_r$ *with piecewise smooth boundary and*

(2) *a piecewise continuously differentiable function* u_r *defined on* M *such that* $u_r = 0$ *on* $F(M)$ *and*

$$\int_M \left(\sum_{i,j=1}^n D_i u_r^T A_{ij}(x) D_j u_r^T - u_r^T C(x) u_r \right) dG \leq 0.$$

Proof. Let $r > 0$ and M be a bounded region in E_r, then the smallest eigenvalue $\lambda(M)$ of the problem

$$Lu + \lambda u = 0 \quad \text{on} \quad M$$
$$u = 0 \quad \text{on} \quad F(M)$$

is less than or equal to zero. Enclose the region M of (i) by a region $M_r(r_1, r_2) = \{x \in E^n \mid r \leq r_1 < |x| < r_2\}$, then the smallest eigenvalue $\lambda(M_r)$ of the region M_r satisfies, $\lambda(M_r) \leq \lambda(M)$. By Theorems B and C there exists a region $M_r' = \{x \in E^n \mid r \leq r_1 < |x| < r_2' \leq r_2\}$ such that the smallest eigenvalue $(M_r') = 0$. Thus the region M_r', is a nodal domain of a non-trivial solution of (1).

Using our methods, a comparison result which gives a stronger result than the results obtained by Kreith [5, 6] and Clark and Swanson [1] can be obtained. Consider equation (1) along with another equation of the same form:

$$L_1 v = \sum_{i,j=1}^{n} D_i (B_{ij}(x) D_j v) + D(x) v = 0. \tag{9}$$

THEOREM 6. *If for some* $R > 0$ *there exists a* $G \subset E_R$ *and a solution* u *of equation* (1) *with* $u = 0$ *on* $F(G)$ *and such that*

$$\int_G [(D_i u^T (A_{ij}(x) - B_{ij}(x)) D_j u + u^T (D(x) - C(x)) u] \, dG \geq 0, \tag{10}$$

then there exists a domain $G' \subset E_R$ *such that* G' *is a nodal domain for a solution* $z(x)$ *of equation* (8).

Proof. Let $G(a, r) = \{x \in E^n \mid 0 < a < |x| < r\}$ be such that $G \subset G(a, r)$. Extend u to $G(a, r)$ by letting $u = 0$ on $\overline{G}(a, r) - G$, since u is a solution of (1) on G and from (10) we have

$$0 = \int_G \left(\sum_{i,j=1}^{n} D_i u^T A_{ij}(x) D_j u - u^T C(x) u \right) dG$$

$$= \int_{G(a,r)} \left(\sum_{i,j=1}^{n} D_i u^T A_{ij}(x) D_j u - u^T C(x) u \right) dG$$

$$\geq \int_{G(a,r)} \left(\sum_{i,j=1}^{n} D_i u^T B_{ij}(x) D_j u - u^T D(x) u \right) dG .$$

Therefore the smallest eigenvalue of the problem

$$L_1 u + \lambda u = 0 \quad \text{on} \quad G(a, r)$$
$$u = 0 \quad \text{on} \quad F(G(a, r))$$

is less than or equal to zero. Again, by Theorems B and C there exists a domain $G' = G(a, r') = \{x \mid 0 < a < |x| < r' \leq r\}$ such that $\lambda_1(G') = 0$.

Another type of comparison theorem involves equation (1) and an equation of the form (3). For prepared solutions of (3), that is, solutions which satisfy $V^* \sum_{j=1}^{n} A_{ij} D_j v$ is symmetric for $i = 1, \cdots, n$, we have the following theorem:

THEOREM 7. *If equation* (1) *is nodally oscillatory, then the determinant of every prepared solution of* (3) *has a zero in* E_R *for every* $R > 0$.

Proof. Since equation (1) is oscillatory, there exists for each $R > 0$ a domain $G \subset E_R$ which is a nodal domain for a solution $u(x)$ of equation (1). Now applying Swanson's Theorem 1 of [11], we get our results.

Note we have assumed the existence of classical solutions of the eigenvalue problem (4). Theorem A guarantees the existence of generalized solutions under the stated assumptions on the coefficients of L.

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