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Fisica matematica. — *Maxwell's equations and Clifford algebra: vector formulation* (*). Nota I di AMALIA ERCOLI FINZI (**) e CARLO MOROSI (**), presentata (***) dal Socio C. CATTANEO.

RIASSUNTO. — Si considera una possibile generalizzazione delle equazioni di Maxwell, nell'ambito del formalismo dell'algebra di Clifford per lo spazio euclideo. Si indaga se tale generalizzazione conservi le proprietà formali della descrizione classica, e si conclude che solo il dato sperimentale della conservazione della carica elettrica e della non esistenza della carica magnetica impone la riduzione al caso classico.

INTRODUCTION

Several and often formally different formulations of the electromagnetic field equations are well-known. Due to their synthetic features, particularly interesting formulations are those representing the field by a unique element of a suitable function space or algebraic set, e.g. by an element of the Clifford algebra: the Pauli algebra for the Euclidean space [1] and the Dirac algebra for the space-time Minkowski manifold [2].

It is known that the field can be described by an element of the Pauli algebra formed only by a vector and a pseudo-vector, and that the charge-current density is given by an element of the so-called even subalgebra of the Pauli algebra, formed by a scalar and a vector [1, p. 16]: therefore one must associate with the physical objects some elements which are not the most general ones of the algebra.

Is such a restricted choice due only to the well-known experimental fact that neither scalar electromagnetic field nor magnetic charge-current distribution exist, or is it implicitly required by some other properties of the field and by the structure of the algebra by which it is described? This paper aims just to verify a possible generalization of the field equations that maintains the same formal structure of the classic equations, but that associates the most general elements of the Pauli algebra with the field and the charge-current density. This analysis seems to us of some interest as the set of equations (*generalized Maxwell's equations*) corresponds to the wave equation with zero rest mass, and the cases of positive definite energy are known for reducible representations (as the one we consider) only for particles with non-zero rest mass [3, ch. 2].

As we will show in the following sections, the conclusion can be drawn that if one requires that also in the generalized field a positive definite energy exists, a vector with the same properties of the Poynting vector can be defined,

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as well as two symmetric stress and stress-energy tensors, then no restrictions have to be imposed upon the generalized field: however, it is characterized by a charge-current density both electric and magnetic, for which no conservation law has to be required. Up to now [1], it can be concluded that the generalized field can be reduced to the classic field only by taking into account the experimental fact of the conservation law for the electric charge and the non-existence of a magnetic charge-current distribution.

1. Mathematical preliminaries

For a concise exposition of the formal aspects of the Clifford algebra and particularly of its physical-geometric meaning, see [1]: in contrast with [1], the terminology of the usual vector calculus will be used for a few notations and definitions. The Pauli algebra $\mathcal{P}$ is used, i.e. the real Clifford algebra C_3 for the Euclidean three-dimensional space. By p-number we mean any element of the algebra (in what follows, any p-number is written in Latin letters): its general form is as follows:

$$p \equiv \alpha + i\beta + \boldsymbol{u} + i\boldsymbol{v} \tag{1.1}$$

that is a combination of a scalar, a pseudo-scalar, a vector and a pseudo-vector [2]. The Clifford product $\boldsymbol{uv}$ of two vectors decomposes as follows

$$\boldsymbol{uv} = \boldsymbol{u} \cdot \boldsymbol{v} + i\boldsymbol{u} \times \boldsymbol{v}, \tag{1.2}$$

where the usual scalar and vector products are indicated by $\cdot$ and $\times$ respectively. If p is a p-number, by $p^\dagger$ we mean its (hermitian) conjugate:

$$p^\dagger \equiv \alpha - i\beta + \boldsymbol{u} - i\boldsymbol{v}. \tag{1.3}$$

When writing differential equations, one has to introduce the differential operator $\partial/\partial\boldsymbol{x}$; it follows from its definition that

$$\frac{\partial\lambda}{\partial\boldsymbol{x}} = \operatorname{grad}\lambda \quad ; \quad \frac{\partial u}{\partial\boldsymbol{x}} = \operatorname{div}\boldsymbol{u} + i\operatorname{rot}\boldsymbol{u}; \tag{1.4}$$

(1) The analysis of the space-time formulation of the generalized equations and their variational formulation is the subject of the second part of this paper (Maxwell's equations and Clifford algebra: space-time formulation. Nota II).

Among other generalized space-time formulations of the electromagnetic field, see [4], where the field is described by a skew-symmetric second order tensor, a scalar function and a completely skew-symmetric tensor of the fourth order.

(2) Scalar objects are written by Greek letters (α, β, $\cdots$), vector objects by Latin bold letters ($\boldsymbol{u}$, $\boldsymbol{v}$, $\cdots$). The unit pseudo-scalar is written by i: as $i^2 = -1$, formally it behaves as the imaginary unit, and it commutes with any p-number ($ip \equiv pi$, $\forall p \in \mathcal{P}$).

therefore by (1.1) it follows that:

$$(1.5) \qquad \frac{\partial p}{\partial x} = \operatorname{div} \boldsymbol{u} + i \operatorname{div} \boldsymbol{v} + (\operatorname{grad} \alpha - \operatorname{rot} \boldsymbol{v}) + i (\operatorname{grad} \beta + \operatorname{rot} \boldsymbol{u}) .$$

The d'Alembertian operator of the Clifford algebra is indicated by

$$\square \equiv \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right);$$

when applied to scalar and vector objects, it behaves as the usual d'Alembertian operator: $\square \equiv \Delta - \partial^2/\partial t^2$.

2. THE CLASSIC VECTOR FORMULATION

If the classic field is considered for the vacuum ($\varepsilon_0 = \mu_0 = c = 1$), and the field is described by $\mathbf{E}, \mathbf{B}$, the electric charge-current density by $\rho, \boldsymbol{j}$, the energy and the Poynting vector by $\varepsilon, \boldsymbol{s}$ ($\varepsilon \equiv (\mathrm{E}^2 + \mathrm{B}^2)/2$; $\boldsymbol{s} \equiv \mathbf{E} \times \mathbf{B}$), the field equations are:

$$(2.1) \qquad \operatorname{div} \mathbf{E} = \rho \quad ; \quad \operatorname{div} \mathbf{B} = 0 \quad ; \quad \frac{\partial \mathbf{E}}{\partial t} - \operatorname{rot} \mathbf{B} = -\boldsymbol{j} \quad ; \quad \frac{\partial \mathbf{B}}{\partial t} + \operatorname{rot} \mathbf{E} = 0 .$$

By means of eqs. (2.1) one can easily obtain the energy equation, the continuity equation and the second order field equations, respectively:

$$(2.2) \qquad \frac{\partial \varepsilon}{\partial t} + \operatorname{div} \boldsymbol{s} = -\mathbf{E} \cdot \boldsymbol{j}; \qquad (2.3) \qquad \frac{\partial \rho}{\partial t} + \operatorname{div} \boldsymbol{j} = 0;$$

$$(2.4') \qquad \square\, \mathbf{E} = \operatorname{grad} \rho + \frac{\partial \boldsymbol{j}}{\partial t}; \qquad (2.4'') \qquad \square\, \mathbf{B} = -\operatorname{rot} \boldsymbol{j} .$$

Furthermore the ponderomotive force law follows from eqs. (2.1):

$$(2.5) \qquad p_{ki}{}^{/k} - \frac{\partial s_i}{\partial t} = \rho \mathrm{E}_i - \varepsilon_{imn} \mathrm{B}^m j^n \text{ (3)}, \qquad (i = 1, 2, 3)$$

where

$$(2.6) \qquad p_{ki} \equiv -\varepsilon \delta_{ik} + \mathrm{E}_i \mathrm{E}_k + \mathrm{B}_i \mathrm{B}_k \qquad (i, k = 1, 2, 3)$$

is the stress tensor. At last, the field $\mathbf{E}, \mathbf{B}$ can be described by means of the potentials $\mathbf{A}, \varphi$ by the well-known relations

$$(2.7) \qquad \mathbf{E} \equiv -\operatorname{grad} \varphi - \frac{\partial \mathbf{A}}{\partial t} \quad ; \quad \mathbf{B} \equiv \operatorname{rot} \mathbf{A} .$$

(3) Here and in the following, the usual notations of tensor calculus, as the summation convention over repeated indices, are adopted.

The potentials can be obtained directly from the charge-current density by the equations

$$\square \mathbf{A} - \mathrm{grad}\, \mathscr{L} = -\boldsymbol{j} \quad ; \quad \square \varphi + \frac{\partial \mathscr{L}}{\partial t} = -\rho \left(\mathscr{L} \equiv \mathrm{div}\, \mathbf{A} + \frac{\partial \varphi}{\partial t}\right), \tag{2.8}$$

or the equations $\square \mathbf{A} = -\boldsymbol{j}$; $\square \varphi = -\rho$, corresponding to the choice of the Lorentz gauge [4].

3. The classic field and the Pauli algebra

As can be easily verified, if the following p-numbers are defined:

$$\mathrm{F} \equiv \mathbf{E} + i\mathbf{B} \quad ; \quad j \equiv \rho - \boldsymbol{j}, \tag{3.1}$$

the field equations (2.1) take the form

$$\frac{\partial \mathrm{F}}{\partial x} + \frac{\partial \mathrm{F}}{\partial t} = j, \tag{3.2}$$

whereas eqs. (2.2) and (2.5) correspond to the unique equation

$$\mathrm{F}^{\dagger}\left(\frac{\partial \mathrm{F}}{\partial x} + \frac{\partial \mathrm{F}}{\partial t}\right) = \mathrm{F}^{\dagger} j, \tag{3.3}$$

(it would be better to say to its scalar and vector parts in $\mathscr{P}$, as the pseudo-scalar and the pseudo-vector parts give only necessary conditions of the field equations (3.2)). Similarly, the conservation and second-order equations (2.4) and (2.3) correspond to the unique equation

$$\square \mathrm{F} = \frac{\partial j}{\partial x} - \frac{\partial j}{\partial t}. \tag{3.4}$$

Furthermore we remark that the description of the field by means of the potentials can be maintained if one defines $f \equiv \varphi - \mathbf{A}$ and assumes the following relation between F and φ

$$\mathrm{F} = \frac{\partial f}{\partial t} - \frac{\partial f}{\partial x}. \tag{3.5}$$

From the vector and pseudo-vector part of eq. (3.5), eq. (2.7) follows, whereas the scalar part gives the Lorentz gauge condition. As the equation $\square f = -j$ yields the equations of the potentials and of the charge-current

(4) The field equations are invariant under gauge transformations of the second kind

$$\mathbf{A} \mapsto \mathbf{A}' \equiv \mathbf{A} - \mathrm{grad}\, \chi \quad ; \quad \varphi \mapsto \varphi' \equiv \varphi - \frac{\partial \chi}{\partial t}.$$

The Lorentz gauge χ ($\square \chi = \mathscr{L}$) corresponds to the only linear gauge transformations invariant under any coordinate mapping [5].

density corresponding to the Lorentz gauge, this gauge is the peculiar one for the Clifford-formalism (as well as for Lorentz-invariant space-time descriptions). Therefore only restricted transformations are allowed

$$f \mapsto f' \equiv f + \frac{\partial \lambda}{\partial t} - \frac{\partial \lambda}{\partial x} \qquad (\Box\, \lambda = 0)\,. \tag{3.6}$$

In contrast with the classic formalism, there is no more any link between gauge invariance and conservation laws: this link depends upon formal properties of the mathematical model more than on physical grounds, in particular it depends upon the adjointness relation between the (formal) operators of the equations corresponding to eqs. (3.2) and (3.5) [6].

4. THE GENERALIZED FIELD

As already said in the introduction, the field equations given in the Clifford formalism are now generalized through the following criterion: we assume that the form of the equations is the same, but that F, f, j, are given by the following more general p-numbers:

$$\mathrm{F}' \equiv \alpha + i\beta + \mathbf{E} + i\mathbf{B} \quad ; \quad f' \equiv \varphi + i\psi - \mathbf{A} - i\mathbf{C}\,; \tag{4.1}$$
$$j' \equiv \rho + i\sigma - \boldsymbol{j} - i\boldsymbol{m}\,.$$

That is to say, the generalized field F′ is given by the most general element of the Pauli algebra, and we suppose that it is either a pure radiation field or it is generated by a general current (as will be seen later on, it is spontaneous to interpret σ and $\boldsymbol{m}$ as a magnetic charge-current density). Equating separately the scalar, vector, pseudo-scalar and pseudo-vector parts in $\mathscr{P}$, we get the four equations:

$$\left\{\begin{aligned} &\frac{\partial \alpha}{\partial t} + \operatorname{div} \mathbf{E} = \rho \quad ; \quad \frac{\partial \beta}{\partial t} + \operatorname{div} \mathbf{B} = \sigma\,; \\ &\frac{\partial \mathbf{B}}{\partial t} + \operatorname{rot} \mathbf{E} + \operatorname{grad} \beta = -\boldsymbol{m} \quad ; \quad \frac{\partial \mathbf{E}}{\partial t} - \operatorname{rot} \mathbf{B} + \operatorname{grad} \alpha = -\boldsymbol{j}\,; \end{aligned}\right. \tag{4.2}$$

the analogy between ρ, σ and $\boldsymbol{j}$, $\boldsymbol{m}$ is self-evident.

If eq. (3.3) is taken into account, its scalar and vector parts give the following equations:

$$\frac{\partial \varepsilon'}{\partial t} + \operatorname{div} \boldsymbol{s}' = -(\mathbf{E} \cdot \boldsymbol{j} + \mathbf{B} \cdot \boldsymbol{m}) + \alpha\rho + \beta\sigma \tag{4.3}$$

$$p'_{ki}{}^{/k} - \frac{\partial s'_i}{\partial t} - \frac{\partial l_i}{\partial t} = \rho \mathrm{E}_i + \sigma \mathrm{B}_i + \varepsilon_{ikn} (j^k\, \mathrm{B}^n - m^k\, \mathrm{E}^n)\,, \tag{4.4}$$

whereas the pseudo-scalar and pseudo-vector parts correspond to necessary (but not sufficient) conditions of the generalized eqs. (4.2). As the following definitions have been introduced in eqs. (4.3) and (4.4)

$$(4.5)\quad \begin{cases} \varepsilon' \equiv (E^2 + B^2 + \alpha^2 + \beta^2)/2 \quad ; \quad \boldsymbol{s}' \equiv \mathbf{E} \times \mathbf{B} + \alpha\mathbf{E} + \beta\mathbf{B}; \\ \boldsymbol{l} \equiv -2\,(\alpha\mathbf{E} + \beta\mathbf{B}); \\ p'_{ki} \equiv (-\varepsilon' + \alpha^2 + \beta^2)\,\delta_{ik} + E_i\,E_k + B_i\,B_k - \varepsilon_{ikn}\,(\alpha B^n - \beta E^n), \end{cases}$$

$\varepsilon', \boldsymbol{s}', p'_{ki}$ appear as the natural generalizations of the classic field energy, Poynting vector and stress tensor; the vector $\boldsymbol{l}$ has no classic analogue and it vanishes for $\alpha = \beta = 0$. The generalized second-order field equations, corresponding to eq. (3.4), are now given by:

$$(4.6)\quad \begin{cases} \Box\,\alpha = -\operatorname{div}\boldsymbol{j} - \dfrac{\partial\rho}{\partial t} \qquad \Box\,\beta = -\operatorname{div}\boldsymbol{m} - \dfrac{\partial\sigma}{\partial t} \\ \Box\,\mathbf{E} = \operatorname{grad}\rho + \dfrac{\partial\boldsymbol{j}}{\partial t} + \operatorname{rot}\boldsymbol{m} \quad ; \quad \Box\,\mathbf{B} = \operatorname{grad}\sigma + \dfrac{\partial\boldsymbol{m}}{\partial t} - \operatorname{rot}\boldsymbol{j}. \end{cases}$$

The relations between field and potentials, and between potentials and current, corresponding to eqs. (3.5) and to $\Box f = -j$, become now:

$$(4.7)\quad \begin{cases} \alpha = \operatorname{div}\mathbf{A} + \dfrac{\partial\varphi}{\partial t} \quad ; \quad \beta = \operatorname{div}\mathbf{C} + \dfrac{\partial\psi}{\partial t} \\ \mathbf{E} = -\left(\dfrac{\partial\mathbf{A}}{\partial t} + \operatorname{grad}\varphi + \operatorname{rot}\mathbf{C}\right) \quad ; \quad \mathbf{B} = -\left(\dfrac{\partial\mathbf{C}}{\partial t} + \operatorname{grad}\psi - \operatorname{rot}\mathbf{A}\right) \end{cases}$$

$$(4.8)\qquad \Box\,\varphi = -\rho \quad ; \quad \Box\,\psi = -\sigma \quad ; \quad \Box\,\mathbf{A} = -\boldsymbol{j} \quad ; \quad \Box\,\mathbf{C} = -\boldsymbol{m}.$$

5. Properties of the generalized field

The formal structure and the properties of the new equations are now analyzed, and their relations with the classic equations discussed. The generalized field eqs. (4.2) are a system of eight partial differential equations of the first order, in normal form with respect to the time variable: therefore one must not expect the existence of compatibility conditions for the equations, i.e. conservation laws for the charge-current density: in fact eq. (4.6) implies that:

$$(5.1)\qquad \operatorname{div}\boldsymbol{j} + \frac{\partial\rho}{\partial t} \not\equiv 0 \quad ; \quad \operatorname{div}\boldsymbol{m} + \frac{\partial\sigma}{\partial t} \not\equiv 0\,.$$

The stress tensor p'_{ki} is not symmetric, but it is well-known [7, ch. 4] that eq. (4.4) can be re-expressed in another equivalent form, where a new symmetric stress tensor is contained, whose linear invariant is the opposite of the (generalized) field energy ε', as in the classic field. To this end, a tensor q_{ki} with zero divergence must be added to p'_{ki}, so that the new tensor becomes symmetric; taking into account the definition (4.5) of p'_{ki}, the following conditions must be satisfied by q_{ki}:

$$(5.2)\qquad q_{ki}^{(s)/k} = V_i\,(V_i \equiv \varepsilon_{imn}\,v^{n|m}) \quad ; \quad q_{ki}^{(e)} = \varepsilon_{kin}\,v^n\,(v^n \equiv \alpha B^n - \beta E^n),$$

where $q_{ki}^{(s)}$ and $q_{ki}^{(e)}$ are the symmetric and skew-symmetric parts of q_{ki}: therefore p'_{ki} can be made symmetric if a symmetric tensor can be obtained with a given divergence: the solution of this problem is known for generic N-manifolds, both Euclidean and Riemannian [8; 9]. Assuming a flat manifold and choosing orthogonal Cartesian coordinates, the general solution of eq. (5.2′) is:

$$q_{ki}^{(s)} = \overset{*}{q}_{ki} + \varepsilon_{kmn}\,\varepsilon_{ilr}\,\chi^{ml/nr} \left(\overset{*}{q}_{kk} = \int^{x^k} \mathrm{V}_k\,\mathrm{d}x^k \; ; \; \overset{*}{q}_{i \neq k} = 0 \right), \tag{5.3}$$

where χ^{ml} is an arbitrary symmetric tensor [10].

Furthermore χ^{ml} may always be chosen so that the linear invariant of the stress be the opposite of ε'; in fact, since

$$p_i'^{\,i} + q_i^{\,i} = -3\,(\varepsilon' - \alpha^2 - \beta^2) + \mathrm{E}^2 + \mathrm{B}^2 + \varepsilon_{imn}\,\varepsilon^i_{lr}\,\chi^{ml/nr} \tag{5.4}$$

it is sufficient that χ^{mn} is a solution of the unique differential equation

$$\Box\,\chi - \chi^{mn}{}_{/mn} = -2\,(\alpha^2 + \beta^2) - \overset{*}{q}{}_k^{k}. \qquad (\chi \equiv \chi^i_i). \tag{5.5}$$

Therefore, if Q_{ki} is the tensor (5.3), where χ^{mn} is any solution of eq. (5.5), the tensor (4.5) can be replaced by the following *symmetric* tensor P_{ki}:

$$\mathrm{P}_{ki} \equiv (-\varepsilon' + \alpha^2 + \beta^2)\,\delta_{ki} + \mathrm{E}_i\,\mathrm{E}_k + \mathrm{B}_i\,\mathrm{B}_k + \mathrm{Q}_{ki}\,. \tag{5.6}$$

The energetic features of the electromagnetic field are classically described by ε, $\boldsymbol{s}$, p_{ki}: the eqs. (2.2) and (2.5) of the time evolution of ε and $\boldsymbol{s}$ can be reunited, in a space-time description (5), by the equation of the ponderomotive force:

$$\mathrm{T}^{\mu\nu}{}_{/\mu} = \mathrm{K}^{\nu}, \tag{5.7}$$

where K^{ν} and the symmetric stress-energy tensor $\mathrm{T}^{\mu\nu}$ are given by

$$\mathrm{K}^{\nu} \equiv (-\,\mathbf{E}\cdot\boldsymbol{j}\,;\,-\,\rho\mathbf{E} + \mathbf{B}\times\boldsymbol{j}) \quad ; \quad \mathrm{T}^{\mu\nu} \equiv \left[\begin{array}{c|c} \varepsilon & \boldsymbol{s} \\ \hline \boldsymbol{s} & -p_{ki} \end{array}\right]. \tag{5.8}$$

Taking into account eqs. (4.3) and (4.4), the generalized form of eq. (5.7) is $\mathrm{T}'^{\mu\nu}{}_{/\mu} = \mathrm{K}'^{\nu}$, with:

$$\mathrm{T}'^{\mu\nu} \equiv \left[\begin{array}{c|c} \varepsilon' & \boldsymbol{s}' + \boldsymbol{l} \\ \hline \boldsymbol{s}' & -\mathrm{P}_{ki} \end{array}\right]. \tag{5.9}$$

The generalized equation can be given in an equivalent form where a symmetric tensor of the second rank $\mathrm{T}''^{\mu\nu}$ appears, containing the energy, a stress tensor and a unique vector.

(5) The metric tensor is $\eta_{\mu\nu} \equiv \mathrm{diag}\,(1\,;-1\,;-1\,;-1)$. Latin indices take on values $(1, 2, 3)$. Greek indices take on values $(0, 1, 2, 3)$.

To this end (and in a substantially analogous way as for the symmetrization of the stress tensor) one searches for a tensor $R^{\mu\nu}$ ($R^{\mu\nu} \equiv T''^{\mu\nu} - T'^{\mu\nu}$) so that:

$$(5.10) \qquad R^{\mu\nu}{}_{/\mu} = 0 ; \qquad\qquad (5.11) \qquad R^{\mu\nu} - R^{\nu\mu} = T'^{\nu\mu} - T'^{\mu\nu} .$$

Eq. (5.10) is satisfied by taking:

$$(5.12) \qquad R^{\mu\nu} \equiv \left[\begin{array}{c|c} \gamma & \boldsymbol{r} - \boldsymbol{l} \\ \hline \boldsymbol{r} & S_{mi} \end{array}\right] ,$$

and eq. (5.11) requires that

$$(5.13) \qquad \begin{cases} \dfrac{\partial\gamma}{\partial t} + \operatorname{div} \boldsymbol{r} = 0 \\ \dfrac{\partial \boldsymbol{r}}{\partial t} - \dfrac{\partial \boldsymbol{l}}{\partial t} + \boldsymbol{u} = 0 \end{cases} \qquad (S^{mi}{}_{/m} \equiv u^i) .$$

Eqs. (5.13) have infinitely many solutions, e.g. we can choose $\boldsymbol{r} = 0$ and $\gamma = 0$ (without loss of generality, for the very meaning of the generalized energy ε'); furthermore, the particular solution S^{*mi} can be chosen of the equation

$$(5.14) \qquad S^{*mi}{}_{/m} = \frac{\partial l^i}{\partial t} ,$$

having a vanishing linear invariant (as mentioned above, this is always possible). Therefore:

$$(5.15) \qquad R^{\mu\nu} \equiv \left[\begin{array}{c|c} 0 & -\boldsymbol{l} \\ \hline 0 & S^{*mi} \end{array}\right] \Rightarrow T''^{\mu\nu} = \left[\begin{array}{c|c} \varepsilon' & \boldsymbol{s}' \\ \hline \boldsymbol{s}' & -P_{mi} + S^{*}_{mi} \end{array}\right] ,$$

$$(5.16) \qquad T''^{\mu\nu}{}_{/\mu} = K'^{\nu} \quad ; \quad T''^{\mu\nu} = T''^{\nu\mu} \quad ; \quad T''^{00} = \varepsilon' > 0 .$$

At last it can be remarked that the generalized Poynting vector $\boldsymbol{s}'$ defined by (4.5) is again parallel to the wave vector $\boldsymbol{k}$ ($j = 0$):

$$(5.17) \qquad \boldsymbol{s}' \times \boldsymbol{k} = (\mathbf{E} \times \mathbf{B}) \times \boldsymbol{k} + \alpha (\mathbf{E} \times \boldsymbol{k}) + \beta (\mathbf{B} \times \boldsymbol{k}) = 0 .$$

In fact eqs. (4.2) have the following *plane wave* solution ($\nu \equiv \boldsymbol{k} \cdot \boldsymbol{x} - \omega t$)

$$(5.18) \qquad \alpha = \alpha_0 \exp(i\nu) \;\; ; \;\; \beta = \beta_0 \exp(i\nu) \;\; ; \;\; \mathbf{E} = \mathbf{E}_0 \exp(i\nu) \;\; ; \;\; \mathbf{B} = \mathbf{B}_0 \exp(i\nu),$$

where, taking into account eqs. (4.2), ω and $\boldsymbol{k}$ satisfy the relations:

$$(5.19) \qquad \begin{cases} \omega\alpha - \boldsymbol{k} \cdot \mathbf{E} = 0 & ; \quad \omega\beta - \boldsymbol{k} \cdot \mathbf{B} = 0 \\ \omega\mathbf{E} + \boldsymbol{k} \times \mathbf{B} - \alpha\boldsymbol{k} = 0 & ; \quad \omega\mathbf{B} - \boldsymbol{k} \times \mathbf{E} - \beta\boldsymbol{k} = 0 . \end{cases}$$

Thus the condition (5.17) can be easily deduced, on account of eqs. (5.19).

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