Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Bang-Yen Chen, Leopold Verstraelen

Surfaces with flat normal connection

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **59** (1975), n.5, p. 407–410.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1975_8_59_5_407_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1975.

Geometria differenziale. — *Surfaces with flat normal connection.* Nota di BANG-YEN CHEN e LEOPOLD VERSTRAELEN, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Dopo aver dato due diverse caratterizzazioni per le superficie di una varietà riemanniana m-dimensionale che hanno una connessione normale piatta, si caratterizzano le superficie sferiche di codimensione 1 e le varietà riemanniane conformemente piatte di dimensione $m > 3$.

§ 1. INTRODUCTION

Let $x : M \to R^m$ be an isometrical immersion of a surface M into an m-dimensional Riemannian manifold R^m and let ∇ and ∇' be the covariant differentiations of M and R^m respectively. Let X and Y be two tangent vector fields on M. Then the second fundamental form h is given by

$$\nabla'_X Y = \nabla_X Y + h(X, Y) . \tag{1}$$

It is well-known that $h(X, Y)$ is a normal vector field on M and is symmetric on X and Y. Let ξ be a normal vector field on M, we write

$$\nabla'_X \xi = -A_\xi(X) + D_X \xi , \tag{2}$$

where $-A_\xi(X)$ and $D_X \xi$ denote the tangential and normal components of $\nabla'_X \xi$. Then D is the *normal connection* of M in R^m and we have

$$\langle A_\xi(X), Y\rangle = \langle h(X, Y), \xi\rangle , \tag{3}$$

where $\langle , \rangle$ denotes the scalar product in R^m. The curvature tensor K^N associated with D is given by

$$K^N(X, Y) = [D_X, D_Y] - D_{[X,Y]} . \tag{4}$$

For a surface M in R^m, if the curvature tensor K^N vanishes identically, then M is said to have *flat normal connection*. For a surface in a conformally flat space, the flatness of normal connection is equivalent to the commutativity of the second fundament tensors [2].

In this note, we shall first obtain two characterizations for surfaces with flat normal connection. Next, we shall apply this to obtain a characterization for spherical surfaces of codimension one. Finally, we shall prove that a Riemannian manifold R^m of dimension $m > 3$ is conformally flat if and only if for any point $p \in R^m$ and any plane section $\pi \subset T_p(R^m)$, there exists a surface M in R^m with $T_p(M) = \pi$ such that the normal connection of M in R^m is flat and the second fundamental tensors commute, where $T_p(M)$ (resp. $T(M)$) is the tangent space of M at p (resp. the tangent bundle of M).

(*) Nella seduta del 15 novembre 1975.

§ 2. CHARACTERIZATIONS OF FLAT NORMAL CONNECTIONS

Let M be a surface in an m-dimensional Riemannian manifold R^m. A normal vector field ξ ($\neq 0$) is called a parallel (resp. umbilical) section if $D\xi = 0$ identically (resp. A_ξ is proportional to the identity transformation). Let X be a vector tangent to M. We denote by $X^\perp$ a vector tangent to M such that $\langle X, X \rangle = \langle X^\perp, X^\perp \rangle$ and $\langle X, X^\perp \rangle = 0$.

LEMMA 1. *Let* M *be a surface in an* **m**-*dimensional Riemannian manifold* R^m. *Then the following three statements are equivalent*:

(a) $[A_\xi, A_\eta] = 0$ *for all normal vectors* ξ, η *at* p;

(b) $\{h(X, X^\perp) : X \in T_p(M)\} \subset$ *line*;

(c) *there exist at least* $m - 3$ *orthogonal sections umbilical at* p.

Proof. (*a*) $\Rightarrow$ (*b*). If the second fundamental tensors commute, there exist an orthonormal basis $\{e_1, e_2\}$ which diagonalize all second fundamental tensors. Hence, we have $h(e_1, e_2) = 0$. Let $X = \Sigma X^i e_i$, $X^\perp = \Sigma Y^j e_j$, $i, j = 1, 2$. Then we have $h(X, X^\perp) = X^1 Y^1 (h(e_1, e_1) - h(e_2, e_2))$.

(*b*) $\Rightarrow$ (*a*) and (*c*). Let $X = \Sigma X^i e_i$, $X^\perp = \Sigma Y^j e_j$, where $\{e_1, e_2\}$ is any orthonormal basis of $T_p(M)$. Then we have

$$h(X, X^\perp) = X^1 Y^1 (h(e_1, e_1) - h(e_2, e_2)) + (X^1 Y^2 + X^2 Y^1) h(e_1, e_2).$$

If $\{h(X, X^\perp) : X \in T_p(M)\} \subset$ line, then $h(e_1, e_2)$ and $h(e_1, e_1) - h(e_2, e_2)$ are linearly dependent. If $h(e_1, e_1) - h(e_2, e_2) = 0$ for any orthonormal basis e_1, e_2, then M is totally umbilical, i.e., every normal vector field is umbilical. Thus, the second fundamental tensors commute. If $h(e_1, e_1) \neq h(e_2, e_2)$ for some orthonormal basis e_1, e_2, then it is clear that every normal vector perpendicular to $h(e_1, e_1) - h(e_2, e_2)$ is umbilical. In particular, all second fundamental tensors commute.

(*c*) $\Rightarrow$ (*a*). This is trivial.

From Lemma A and Theorem 4 of [2], we have immediately the following

THEOREM 1. *Let* M *be a surface in an* m-*dimensional conformally flat space* R^m ($m > 3$). *Then the normal connection of* M *in* R^m *is flat if and only if one of the following three conditions holds*:

(a) $\dim \{h(X, X^\perp) : X \in T_p(M)\} \leqq 1$ *for all* $p \in M$;

(b) *there exist at least* $m - 3$ *orthogonal umbilical sections*;

(c) *second fundamental tensors commute*.

REMARK 1. For results in this direction, see also [3, 6].

§ 3. CHARACTERIZATION OF "SPHERICAL" SURFACES AND CONFORMALLY FLAT SPACES

Following [1], by a space form $R^m(k)$ of curvature k, we mean a complete simply-connected Riemannian manifold of constant sectional curvature k. By an n-sphere of $R^m(k)$ we mean a hypersphere of an $(n+1)$-dimensional totally geodesic submanifold of $R^m(k)$.

If M is a surface in a 3-sphere S^3 of a space form $R^m(k)$, then the normal connection of M in $R^m(k)$ is flat, $\{h(X, X^\perp) : X \in T(M)\}$ is parallel to the normal vector of M in S^3 and gives a parallel section in $R^m(k)$. Conversely, we have the following

THEOREM 2. *Let* M *be a surface in an m-dimensional space form* $R^m(k)$. *If the normal connection of* M *in* $R^m(k)$ *is flat and* $\{h(X, X^\perp) : X \in T(M)\}$ *gives a parallel (normal) section in* $R^m(k)$, *then* M *lies in a 3-sphere of* $R^m(k)$.

Proof. Since the normal connection of M in $R^m(k)$ is flat, there exists locally an orthonormal basis $\{e_1, e_2\}$ of T(M) such that $h(e_1, e_2) = 0$. Let $\xi_1, \cdots, \xi_{m-2}$ be orthonormal normal vector fields such that ξ_1 is parallel to $\{h(X, X^\perp) : X \in T(M)\}$. Then by the assumption, $D\xi_1 = 0$, and $\xi_2, \cdots, \xi_{m-2}$ are umbilical sections. If $A_2 = \cdots = A_{m-2} = 0$, $A_\alpha = A_{\xi_\alpha}$, then M is contained in a 3-dimensional totally geodesic submanifold of $R^m(k)$ [4]. Hence M lies a great 3-sphere of $R^m(k)$. If $A_2, \cdots, A_{m-2}$ are not all zero, then we may choose $\xi_2, \cdots, \xi_{m-2}$ in such a way that $A_2 = \lambda I$, $A_3 = \cdots = A_{m-2} = 0$. Thus, by the following equation of Codazzi:

$$(\nabla_Y A_\xi)(X) + A_{D_X\xi}(Y) = (\nabla_X A_\xi)(Y) + A_{D_Y\xi}(X),$$

and the equation $D\xi_1 = 0$, we find

$$(Y\lambda) X = (X\lambda) Y, \tag{5}$$

$$A_{D_X\xi_\alpha}(Y) = A_{D_Y\xi_\alpha}(X), \qquad \alpha = 3, \cdots, m-2. \tag{6}$$

From (5) and (6) we see that λ is constant and ξ_2 is parallel. Thus M lies in a small 3-sphere of $R^m(k)$ [1].

If R^m is a conformally flat space, then it is clear that for any point $p \in R^m$ and any plane section $\pi \subset T_p(R^m)$ there exists a surface M in R^m through p, tangent to π, with flat normal connection and commutative second fundamental tensors. In the following, we shall prove that the converse of this is also true.

THEOREM 3. *An m-dimensional* $(m > 3)$ *Riemannian manifold* R^m *is conformally flat if an only if for every point* $p \in M$ *and any plane section* $\pi \subset T_p(R^m)$ *there exists a surface in* R^m *through* p, *tangent to* π, *with flat normal connection and commutative second fundamental tensors.*

Proof. We need only to prove the converse. Let p be any point in R^m and X and Y be any two orthonormal vectors in $T_p(R^m)$. Let π be the plane section in $T_p(R^m)$ containing X and Y. Then, by the hypothesis, there exists a surface through p, tangent to M, with flat normal connection and commutative second fundamental tensors. Let ξ and η be any two orthonormal normal vector field of M in R^m. Then we have

$$\langle K^N(X, Y)\xi, \eta\rangle = \langle [A_\xi, A_\eta](X), Y\rangle = 0 .$$

Substituting this into the equation of Ricci ([1], p. 47), we find $\langle \tilde{\tilde{K}}(X, Y)\xi, \eta\rangle = 0$ where $\tilde{\tilde{K}}$ is the curvature tensor of R^m. Since this is true for all points $p \in M$ and all orthonormal vectors X, Y, ξ, η in $T_p(R^m)$, R^m must be conformally flat ([5], p. 307).

REFERENCES

[1] B.-Y. CHEN (1973) – *Geometry of Submanifolds*, M. Dekker, New York.

[2] B.-Y. CHEN (1974) – *Some conformal invariants of submanifolds and their applications*, « Boll. U.M.I. » (4), *10*, 380–385.

[3] B.-Y. CHEN (1974) – *Some results for surfaces with flat normal connection*, « Atti Acad. Naz. Lincei », *56*, 180–188.

[4] J. A. ERBACHER (1971) – *Reducation of the codimension of an isometric immersion*, « J. Differential Geometry », *5*, 333–340.

[5] J. A. SCHOUTEN (1954) – *Ricci-Calculus*, Springer, Berlin.

[6] L. VERSTRAELEN – *On surfaces with flat normal connection in a 4-dimensional elliptic space*, to appear.