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Fixed point mappings

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Geometria differenziale. — *Fixed point mappings.* Nota di BRIAN FISHER, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si dimostra che, se T denota un'applicazione di uno spazio metrico completo X in sè che soddisfi alla

$$\rho(Tx, Ty) \leq a\rho(x, y) + b[\rho(x, Tx) + \rho(y, Ty)] + c[\rho(x, Ty) + \rho(y, Tx)]$$

per tutti gli x, y di X ed a, b, c numeri reali soddisfacenti alle

$$0 \leq \frac{a+b+c}{1-b-c} < 1, \quad b+c, a+2c < 1, c \geq 0,$$

allora T ammette uno ed un solo punto fisso.

The following theorem is the well-known contraction mapping theorem:

THEOREM 1. *If* T *is a mapping of a complete metric space* X *into itself satisfying the condition*

$$\rho(Tx, Ty) \leq a\rho(x, y)$$

for all x, y *in* X, *where* $0 \leq a < 1$, *then* T *has a unique fixed point.*

In a paper by R. Kannan [2] he proved the following theorem:

THEOREM 2. *If* T *is a mapping of the complete metric space* X *into itself satisfying the condition*

$$\rho(Tx, Ty) \leq b[\rho(x, Tx) + \rho(y, Ty)]$$

for all x, y *in* X, *where* $0 \leq b < \frac{1}{2}$, *then* T *has a unique fixed point.*

In [1], the following theorem was proved:

THEOREM 3. *If* T *is a mapping of the complete metric space* X *into itself satisfying the condition*

$$\rho(Tx, Ty) \leq c[\rho(x, Ty) + \rho(y, Tx)]$$

for all x, y *in* X, *where* $0 \leq c < \frac{1}{2}$, *then* T *has a unique fixed point.*

We will now prove the following theorem which includes each of these three theorems as special cases:

THEOREM 4. *If* T *is a mapping of the complete metric space* X *into itself satisfying the condition*

$$\rho(Tx, Ty) \leq a\rho(x, y) + b[\rho(x, Tx) + \rho(y, Ty)] + c[\rho(x, Ty) + \rho(y, Tx)]$$

(*) Nella seduta del 15 novembre 1975.

for all x , y *in* X *where*

$$0 \leq \frac{a+b+c}{1-b-c} < 1 , \quad b+c , a+2c < 1 , c \geq 0 ,$$

then T *has a unique fixed point.*

Proof: Let x be an arbitrary point in X. Then

$$\begin{aligned} \rho(T^n x , T^{n+1} x) &\leq a\rho(T^{n-1} x , T^n x) + b[\rho(T^{n-1} x , T^n x) + \\ &+ \rho(T^n x , T^{n+1} x)] + c[\rho(T^{n-1} x , T^{n+1} x) + \rho(T^n x , T^n x)] \\ &\leq (a+b)\rho(T^{n-1} x , T^n x) + b\rho(T^n x , T^{n+1} x) + c[\rho(T^{n-1} x , T^n x) + \\ &+ \rho(T^n x , T^{n+1} x)] = (a+b+c)\rho(T^{n-1} x , T^n x) + (b+c)\rho(T^n x , T^{n+1} x). \end{aligned}$$

It follows that

$$\begin{aligned} \rho(T^n x , T^{n+1} x) &\leq \frac{a+b+c}{1-b-c}\rho(T^{n-1} x , T^n x) \\ &\leq \left(\frac{a+b+c}{1-b-c}\right)^2 \rho(T^{n-2} x , T^{n-1} x) \\ &\leq \left(\frac{a+b+c}{1-b-c}\right)^n \rho(x , Tx) . \end{aligned}$$

Hence

$$\begin{aligned} \rho(T^n x , T^{n+r} x) &\leq \rho(T^n x , T^{n+1} x) + \cdots + \rho(T^{n+r-1} x , T^{n+r} x) \leq \\ &\leq \left[\left(\frac{a+b+c}{1-b-c}\right)^n + \cdots + \left(\frac{a+b+c}{1-b-c}\right)^{n+r-1}\right]\rho(x , Tx) \leq \\ &\leq \left(\frac{a+b+c}{1-b-c}\right)^n \frac{1-b-c}{1-a-2b-2c}\rho(x , Tx) . \end{aligned}$$

Since

$$\frac{a+b+c}{1-b-c} < 1$$

it follows that $\{T^n x\}$ is a Cauchy sequence in X and so has a limit z in X, since X is complete.

We now have

$$\begin{aligned} \rho(z , Tz) &\leq \rho(z , T^n x) + \rho(T^n x , Tz) \leq \rho(z , T^n x) + a\rho(T^{n-1} x , z) + \\ &+ b[\rho(T^{n-1} x , T^n x) + \rho(z , Tz)] + c[\rho(T^{n-1} x , Tz) + \rho(z , T^n x)\}. \end{aligned}$$

Letting n tend to infinity we see that

$$\rho(z , Tz) \leq (b+c)\rho(z , Tz)$$

and, since $b+c<1$, it follows that

$$Tz = z.$$

Hence z is a fixed point.

Now suppose T has a second fixed point z'. Then

$$\begin{aligned}\rho(z, z') &= \rho(Tz, Tz') \\ &\leq a\rho(z, z') + b[\rho(z, Tz) + \rho(z', Tz')] + c[\rho(z, Tz') + \rho(z', Tz)] = \\ &= (a + 2c)\rho(z, z')\end{aligned}$$

and, since $a + 2c < 1$, it follows that $z = z'$. Hence the fixed point is unique. This completes the proof of the theorem.

We note that if $b = c = 0$, then $0 \leq a < 1$ which gives Theorem 1; if $a = c = 0$, then $0 \leq b < \frac{1}{2}$ which gives Theorem 2; and if $a = b = 0$, then $0 \leq c < \frac{1}{2}$ which gives Theorem 3.

We finally note that a, b, c are not restricted to the values given in Theorems 1, 2 and 3. For example, we could have $a = -3/4$, $b = 1/16$, $c = 3/4$ or $a = 1/3$, $b = -1/4$, $c = 1/4$. In fact it can be shown quite easily that a, b, c can take values in the ranges

$$-1 < a < 1 \quad , \quad -\tfrac{1}{2} < b < 1 \quad , \quad 0 \leq c < 1 .$$

REFERENCES

[1] B. FISHER – *A fixed point theorem*, «Mathematics Magazine», *48*, 223–5.
[2] R. KANNAN (1968) – *Some results on fixed points*, « Bull. Calcutta Math. Soc. », *60*, 71–6.