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A note on cohomology groups on Kaehler manifolds


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Riassunto. — Si dimostra una proprietà dello spazio delle forme armoniche su di una varietà kähleriana, e da essa si traggono alcune conseguenze.

Let $M$ be a compact Kaehler manifold. Let $\Omega^p$ be the sheaf of germs of holomorphic tangent vector fields on $M$. The $q$-th cohomology group of $M$ with coefficients in $\Omega^p$ is isomorphic to the space $H^{p,q}(M, C)$ of “harmonic” $(p, q)$-forms on $M$, i.e. the complex vector space consisting of those $C^\infty$ forms $\varphi$ of type $(p, q)$ such that $\Box \varphi = 0$ on $M$, where $\Box$ is the Laplace-Beltrami operator for the Kaehler metric on $M$.

If $w$ denotes the exterior form of the Kaehler metric on $M$, then, for every positive integer $p \leq n$ (where $n = \dim M$), the $(p, q)$-form $w^p = w \wedge \cdots \wedge w$ belongs to $H^{p,p}(M, C)$; thus dim $H^{p,p}(M, C) = 0$.

Let $C^{r,s}$ be the space of $C^\infty (r, s)$-forms on $M$. Let $L: C^{r,s} \to C^{r+1,s+1}$ be the operator defined locally by $L\varphi = w \wedge \varphi$ (where $\wedge$ denotes the exterior product) and let $\Lambda$ be its adjoint with respect to the euclidean structure defined by the Kaehler metric; $\Lambda$ is expressed by $\Lambda = (-1)^{r+s} L^*$, where $*: C^{r,s} \to C^{n-s,n-r}$ is the operator defined by the Kaehler metric.

The object of this paper is to prove the following

**Theorem.** Let $M$ be a compact, Kaehler manifold of (complex) dimension $n$ and let $p$ be an odd number, $0 \leq p \leq n/2$. If $\dim H^{p,p}(M, C) = k$ and $k \leq (n-p)/p$, then

$$H^{r,s}(M, C) \wedge \cdots \wedge H^{r,s}(M, C) = 0,$$

for every pair $(r, s)$ such that $r + s = p$.

**Proof.** The scalar product $(\varphi, \psi) = \int_M \varphi \Lambda^* \psi (\varphi, \psi \in H^{p,p}(M, C))$

defines on $H^{p,p}(M, C)$ the structure of a finite dimensional Hilbert space. Note that $(w^p, w^p) = 1$. Let $\varphi_1, \ldots, \varphi_k$ be elements of $H^{p,p}(M, C)$ which, with $w^p$, form an orthonormal system in $H^{p,p}(M, C)$.

Let $\varphi_1, \ldots, \varphi_k$ be forms of type $(r, s)$ in $H^{r,s}(M, C)$, then $\varphi_1, \ldots, \varphi_k \in H^{r,s}(M, C)$, since $\Box$ is real: $\Box \varphi_i = \Box \varphi_i = 0$. Consider the $(p, q)$-forms $\varphi_i \wedge \varphi_i$ for $i = 1, \ldots, k$; they satisfy the conditions

$$3(\varphi_i \wedge \varphi_i) = 0 \quad \text{and} \quad 3(\varphi_i \wedge \varphi_i) = 0.$$

(*) Nella seduta del 15 novembre 1975.
Then, we can write, for each $i = 1, \ldots, k$:

$$\varphi_i \wedge \bar{\varphi}_i = \lambda_{i1} \omega^p + \sum_{j=2}^{k} \lambda_{ij} \Omega_j + \bar{\omega}^j,$$

where $\omega^j$ is a $(\rho, \rho - 1)$-form and $\lambda_{ij}$ are complex numbers. Since $\Lambda^p(\varphi_i \wedge \bar{\varphi}_i)$ is a non negative continuous function on $M$ (it is zero only where $\varphi_i$ is zero) and since

$$0 = \left( \sum_{j=2}^{k} \lambda_{ij} \Omega_j + \bar{\omega}^j, \omega^p \right) = \int_M \Lambda^p \left( \sum_{j=2}^{k} \lambda_{ij} \Omega_j + \bar{\omega}^j \right) dv,$$

where $dv$ is the volume element of $M$, then we get:

$$0 < \int_M \Lambda^p (\varphi_i \wedge \bar{\varphi}_i) dv = \int_M \lambda_{i1} (\Lambda^p \omega^p) dv.$$

Therefore, we can suppose that $\lambda_{i1} = 1$, for each $i = 1, \ldots, k$. Let us consider now the $k \times k$ complex matrix

$$\Gamma = \begin{pmatrix}
1 & \cdots & 1 \\
\lambda_{i2} & \cdots & \lambda_{ik} \\
\vdots & & \vdots \\
\lambda_{ik} & \cdots & \lambda_{kk}
\end{pmatrix}.$$

We show that $\det \Gamma = 0$. If this were not the case, given a vector $B \in \mathbb{C}^k$, of the type $B = (c, 0, \ldots, 0)$, $c \neq 0$, we could find a vector $A = (a_1, \ldots, a_k) \in \mathbb{C}^k$ such that: $\Gamma A = B$. Then, we should get:

$$\Sigma_i a_i \varphi_i \wedge \bar{\varphi}_i = c \omega^p + \bar{\omega} (\Sigma_i a_i h_i)$$

and $(\Sigma_i a_i \varphi_i \wedge \bar{\varphi}_i)^{k+1} = 0 = c^{k+1} \omega^{p(k+1)} + (\bar{\omega} (\Sigma_i a_i h_i))^{k+1}$, which is absurd if $c \neq 0$ and $p (k + 1) \leq n$, since $(\bar{\omega} (\Sigma_i a_i h_i))^{k+1} \in \text{Image } \bar{\omega}$. Then $\det \Gamma = 0$, and there is a non zero vector in $\mathbb{C}^k$, $(\beta_1, \ldots, \beta_k)$, such that

$$\Sigma_i \beta_i \varphi_i \wedge \bar{\varphi}_i \in \text{Im } \bar{\omega}.$$

Suppose that $\beta_i = 0$, for $i = 1, \ldots, t, t \leq k$; then

$$\Sigma_{i=1}^{t} \beta_i \varphi_i \wedge \bar{\varphi}_i \in \text{Im } \bar{\omega},$$

$$\left( \Sigma_{i=1}^{t} \beta_i \varphi_i \wedge \bar{\varphi}_i \right) = t ! \beta_1 \cdots \beta_t \cdot \varphi_1 \wedge \bar{\varphi}_1 \wedge \cdots \wedge \varphi_t \wedge \bar{\varphi}_t \in \text{Im } \bar{\omega}.$$

So, since $\beta_1 \cdots \beta_t = 0$, we obtain:

$$(\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_k) \wedge (\bar{\varphi}_1 \wedge \bar{\varphi}_2 \wedge \cdots \wedge \bar{\varphi}_k) \in \text{Im } \bar{\omega},$$

and

$$\psi \wedge \bar{\psi} \in \text{Im } \bar{\omega}, \text{ where } \psi = \varphi_1 \wedge \cdots \wedge \varphi_t.$$
Since $p \cdot t \leq n$, we have

$$\int_M \Lambda^p (\psi \wedge \overline{\psi}) \, dv = (\psi \wedge \overline{\psi}, \omega^p) = 0,$$

which implies that $\psi (x) = 0$, by the same argument as before for $\varphi_i$.

Remark. In the proof of the above theorem, we did not really use the fact that $\varphi_i \in H^{r,s} (M, \mathbb{C})$, but only that $\varphi_i = 0$ and $\overline{\varphi}_i = 0$. So, we could state the following result: if $\dim H^{p,p} (M, \mathbb{C}) = k$, with $0 < p \leq n/2$, $p$ odd and $k \leq (n-p)/p$, then whenever $\varphi_1, \ldots, \varphi_k$ are $(r,s)$-forms with $r+s=p$ and $\varphi_i = 0$, $\overline{\varphi}_i = 0$ for each $i = 1, \ldots, k$, it holds: $\varphi_1 \wedge \cdots \wedge \varphi_k = 0$.

**Corollary 1.** Let $M$ be a compact, Kahler manifold of dimension $n$. Then:

i) if $0 < p \leq n/2$, $p$ odd and $\dim H^{p,p} (M, \mathbb{C}) = 1$, then $H^{r,s} (M, \mathbb{C}) = 0$ for each pair $(r,s)$ with $r+s \leq p$ and $(r+s)$ odd.

ii) if $n/2 \leq l < n$, $(n-l)$ odd and $\dim H^{l,l} (M, \mathbb{C}) = 1$, then $H^{r,s} (M, \mathbb{C}) = 0$ for each pair $(r,s)$ with $r+s \leq n+l$ and $(r+s)$ odd.

Proof. i) follows easily from the above theorem, in view of the injectivity of the operator $L$ on forms of type $(a,b)$ with $a+b \leq n+1$ (see [1]).

ii) follows from i), when we take $p = n-l$ and we note that $H^{r,s} (M, \mathbb{C}) \simeq H^{n-r,n-s} (M, \mathbb{C})$.

**Corollary 2.** If $n$ is even and $p$ is odd with $0 < p \leq n/2$, and if $\dim H^{p,p} (M, \mathbb{C}) = 1$, then $H^{r,s} (M, \mathbb{C}) = 0$ for each pair $(r,s)$ such that $r+s$ is odd and either $r+s \leq p$ or $r+s \geq 2n-p$.

The proof follows from Corollary 1.

Denoting by $B_v$ the $v$-th Betti number of $M$, we have

**Corollary 3.** If $B_{2(2v+1)} = 1$ with $2v+1 \leq n/2$, then $B_{2v+1} = 0$.

**Bibliography**
