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A note on cohomology groups on Kaehler manifolds

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Geometria differenziale. — *A note on cohomology groups on Kaehler manifolds.* Nota di GIULIANA GIGANTE, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si dimostra una proprietà dello spazio delle forme armoniche su di una varietà kähleriana, e da essa si traggono alcune conseguenze.

Let M be a compact Kaehler manifold. Let Ω^p be the sheaf of germs of holomorphic tangent vector fields on M. The q-th cohomology "group" of M with coefficients in Ω^p is isomorphic to the space $H^{p,q}(M, \mathbf{C})$ of "harmonic" (p, q)-forms on M, i.e. the complex vector space consisting of those C^∞ forms φ of type (p, q) such that $\square\varphi = 0$ on M, where $\square$ is the Laplace-Beltrami operator for the Kaehler metric on M.

If w denotes the exterior form of the Kaehler metric on M, then, for every positive integer $p \leq n$ ($n = \dim$ M), the (p, p)-form $w^p = w \wedge \cdots \wedge w$ belongs to $H^{p,p}(M, \mathbf{C})$; thus $\dim H^{p,p}(M, \mathbf{C}) = 0$.

Let $C^{r,s}$ be the space of C^∞ (r, s)-forms on M. Let $L : C^{r,s} \to C^{r+1,s+1}$ be the operator defined locally by $L\varphi = w \wedge \varphi$ (where $\wedge$ denotes the exterior product) and let Λ be its adjoint with respect to the euclidean structure defined by the Kaehler metric; Λ is expressed by $\Lambda = (-1)^{r+s} * L *$, where $* : C^{r,s} \to C^{n-s,n-r}$ is the operator defined by the Kaehler metric.

The object of this paper is to prove the following

THEOREM. *Let* M *be a compact, Kaehler manifold of (complex) dimension* n *and let* p *be an odd number,* $0 \leq p \leq n/2$. *If* $\dim H^{p,p}(M, \mathbf{C}) = k$ *and* $k \leq (n-p)/p$, *then*

$$H^{r,s}(M, \mathbf{C}) \underset{(k\text{-times})}{\wedge \cdots \wedge} H^{r,s}(M, \mathbf{C}) = 0,$$

for every pair (r, s) *such that* $r + s = p$.

Proof. The scalar product $(\varphi, \psi) = \int_M \varphi \Lambda * \bar{\psi}$ $(\varphi, \psi \in H^{p,p}(M, \mathbf{C}))$ defines on $H^{p,p}(M, C)$ the structure of a finite dimensional Hilbert space. Note that $(w^p, w^p) = 1$. Let $\Omega_2, \cdots, \Omega_k$ be elements of $H^{p,p}(M, \mathbf{C})$ which, with w^p, form an orthonormal system in $H^{p,p}(M, \mathbf{C})$.

Let $\varphi_1, \cdots, \varphi_k$ be forms of type (r, s) in $H^{r,s}(M, \mathbf{C})$, then $\bar{\varphi}_1, \cdots, \bar{\varphi}_k \in H^{r,s}(M, \mathbf{C})$, since $\square$ is real: $\square\bar{\varphi}_i = \overline{\square\varphi_i} = 0$. Consider the (p, p)-forms $\varphi_i \wedge \bar{\varphi}_i$ for $i = 1, \cdots, k$; they satisfy the conditions

$$\partial(\varphi_i \wedge \bar{\varphi}_i) = 0 \qquad \text{and} \qquad \bar{\partial}(\varphi_i \wedge \bar{\varphi}_i) = 0.$$

(*) Nella seduta del 15 novembre 1975.

Then, we can write, for each $i = 1, \cdots, k$:

$$\varphi_i \wedge \overline{\varphi}_i = \lambda_{i1} w^p + \sum_2^k{}_j \lambda_{ij} \Omega_i + \bar{\partial} h_i ,$$

where h_i is a $(p, p - 1)$-form and λ_{ij} are complex numbers. Since $\Lambda^p (\varphi_i \wedge \overline{\varphi}_i)$ is a non negative continuous function on M (it is zero only where φ_i is zero) and since

$$0 = \left(\sum_2^k{}_j \lambda_{ij} \Omega_i + \bar{\partial} h_i , w^p \right) = \int_M \Lambda^p \left(\sum_2^k{}_j \lambda_{ij} \Omega_j + \bar{\partial} h_i \right) dv ,$$

where dv is the volume element of M, then we get:

$$0 < \int_M \Lambda^p (\varphi_i \wedge \overline{\varphi}_i) \, dv = \int_M \lambda_{i1} (\Lambda^p w^p) \, dv.$$

Therefore, we can suppose that $\lambda_{i1} = 1$, for each $i = 1, \cdots, k$. Let us consider now the $k \times k$ complex matrix

$$\Gamma = \begin{pmatrix} 1 & \cdots & 1 \\ \lambda_{12} & \cdots & \lambda_{k2} \\ \cdots & \cdots & \cdots \\ \lambda_{1k} & \cdots & \lambda_{kk} \end{pmatrix} .$$

We show that $\det \Gamma = 0$. If this were not the case, given a vector $B \in \mathbf{C}^k$, of the type $B = {}^t(c, 0, \cdots, 0)$, $c \neq 0$, we could find a vector $A = {}^t(\alpha_1, \cdots, \alpha_k) \in \mathbf{C}^k$ such that: $\Gamma A = B$. Then, we should get:

$$\Sigma_i \alpha_i \varphi_i \wedge \overline{\varphi}_i = c w^p + \bar{\partial} (\Sigma_i \alpha_i h_i)$$

and $(\Sigma_i \alpha_i \varphi_i \wedge \overline{\varphi}_i)^{k+1} = 0 = c^{k+1} w^{p(k+1)} + (\bar{\partial} (\Sigma_i \alpha_i h_i))^{k+1}$, which is absurd if $c \neq 0$ and $p(k+1) \leq n$, since $(\bar{\partial} (\Sigma_i \alpha_i h_i))^{k+1} \in \operatorname{Image} \bar{\partial}$. Then $\det \Gamma = 0$, and there is a non zero vector in $\mathbf{C}^k$, $(\beta_1, \cdots, \beta_k)$, such that

$$\Sigma_i \beta_i \varphi_i \wedge \overline{\varphi}_i \in \operatorname{Im} \bar{\partial} .$$

Suppose that $\beta_i \neq 0$, for $i = 1, \cdots, t$, $t \leq k$; then

$$\sum_1^t{}_i \beta_i \varphi_i \wedge \overline{\varphi}_i \in \operatorname{Im} \bar{\partial},$$

$$\left(\sum_1^t{}_i \beta_i \varphi_i \wedge \overline{\varphi}_i \right)^t = t! \, \beta_1 \cdots \beta_t \cdot \varphi_1 \wedge \overline{\varphi}_1 \wedge \cdots \wedge \varphi_t \wedge \overline{\varphi}_t \in \operatorname{Im} \bar{\partial}.$$

So, since $\beta_1 \cdots \beta_t \neq 0$, we obtain:

$$(\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_t) \wedge (\overline{\varphi}_1 \wedge \overline{\varphi}_2 \wedge \cdots \wedge \overline{\varphi}_t) \in \operatorname{Im} \bar{\partial} ,$$

and

$$\psi \wedge \overline{\psi} \in \operatorname{Im} \bar{\partial} , \quad \text{where } \psi = \varphi_1 \wedge \cdots \wedge \varphi_t .$$

Since $p \cdot t \leq n$, we have

$$\int_M \Lambda^{pt} (\psi \wedge \bar{\psi}) \, dv = (\psi \wedge \bar{\psi}, w^{pt}) = 0 ,$$

which implies that $\psi(x) = 0$, by the same argument as before for φ_i.

Remark. In the proof of the above theorem, we did not really use the fact that $\varphi_i \in H^{r,s}(M, \mathbf{C})$, but only that $\partial\varphi_i = 0$ and $\bar{\partial}\varphi_i = 0$. So, we could state the following result: *if* $\dim H^{p,p}(M, \mathbf{C}) = k$, *with* $0 < p \leq n/2$, p *odd and* $k \leq (n - p)/p$, *then whenever* $\varphi_1, \cdots, \varphi_k$ *are* (r, s)*-forms with* $r + s = p$ *and* $\partial\varphi_i = 0$, $\bar{\partial}\varphi_i = 0$ *for each* $i = 1, \cdots, k$, *it holds*: $\varphi_1 \wedge \cdots \wedge \varphi_k = 0$.

COROLLARY 1. *Let* M *be a compact, Kaehler manifold of dimension* n. *Then*:

i) if $0 < p \leq n/2$, p *odd and* $\dim H^{p,p}(M, \mathbf{C}) = 1$, *then* $H^{r,s}(M, \mathbf{C}) = 0$ *for each pair* (r, s) *with* $r + s \leq p$ *and* $(r + s)$ *odd.*

ii) *if* $n/2 \leq l < n$, $(n - l)$ *odd and* $\dim H^{l,l}(M, \mathbf{C}) = 1$, *then* $H^{r,s}(M, \mathbf{C}) = 0$ *for each pair* (r, s) *with* $r + s \leq n + l$ *and* $(r + s)$ *odd.*

Proof. i) follows easily from the above theorem, in view of the injectivity of the operator L on forms of type (a, b) with $a + b \leq n + 1$ (see [1]).

ii) follows from i), when we take $p = n - l$ and we note that $H^{r,s}(M, \mathbf{C}) \simeq H^{n-r,n-s}(M, \mathbf{C})$.

COROLLARY 2. *If* n *is even and* p *is odd with* $0 < p \leq n/2$, *and if* $\dim H^{p,p}(M, \mathbf{C}) = 1$, *then* $H^{r,s}(M, \mathbf{C}) = 0$ *for each pair* (r, s) *such that* $r + s$ *is odd and either* $r + s \leq p$ *or* $r + s \geq 2n - p$.

The proof follows from Corollary 1.

Denoting by B_ν the ν-th Betti number of M, we have

COROLLARY 3. *If* $B_{2(2\nu+1)} = 1$ *with* $2\nu + 1 \leq n/2$, *then* $B_{2\nu+1} = 0$.

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