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Comparison and oscillation theorems for singular hyperbolic equations

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Equazioni a derivate parziali. — *Comparison and oscillation theorems for singular hyperbolic equations.* Nota di EUTIQUIO C. YOUNG, presentata (*) dal Socio M. PICONE.

RIASSUNTO. — In questa Nota si dimostrano teoremi di confronto e di oscillazione per equazioni singolari a derivate parziali del tipo iperbolico considerato in un dominio cilindrico. È notevole uno dei risultati conseguiti, secondo il quale, una soluzione di una tale equazione che sia identicamente nulla sulla superficie laterale di un cilindro, ha nell'interno infiniti punti di zero.

1. INTRODUCTION

The Sturmian comparison and oscillation theorems for ordinary differential equations have been extended extensively to partial differential equations of the elliptic type. For example, see Kuks [1], Swanson [2], [3], Diaz and McLaughlin [4], and Kreith and Travis [5], to mention only a few. In [6], by employing Swanson's technique, Dunninger obtained a comparison theorem for parabolic partial differential equations. His result was recently generalized by Chan and Young [7] to time-dependent quasilinear differential systems. However, for partial differential equations of hyperbolic type very little is known. As a matter of fact, certain Sturmian results for elliptic equations are false for hyperbolic equations without additional constraints. A simple example is the wave equation $u_{tt} - u_{xx} = 0$ in the semi-infinite strip $S = \{(x, t) \mid 0 \leq x \leq \pi, 0 \leq t < \infty\}$. Clearly $u(x, t) = \sin x \sin t$ is a solution of the wave equation which has infinitely many zeros in S while $u(x, t) = 1$ is also a solution which has no zero in S.

In [8] Kreith proved comparison and oscillation theorems for solutions of an initial boundary value problem for the damped wave equation in two variables. His results have been recently extended by Travis [9] to the normal hyperbolic equation in n space variables. Other oscillation results for solutions of hyperbolic equations have also been obtained by Kahane [10] under somewhat different conditions. The purpose of this paper is to present corresponding results for the pair of singular hyperbolic differential equations

$$Lu \equiv u_{tt} + \frac{k}{t} u_t - (a_{ij}(x, t) u_{x_i})_{x_j} + p(x, t) u = 0$$

and

$$Mv \equiv v_{tt} + \frac{k}{t} v_t - (b_{ij}(x, t) v_{x_i})_{x_j} + q(x, t) v = 0$$

where k is a real parameter $-\infty < k < \infty$, and the repeated indices are to be summed from 1 to n.

(*) Nella seduta del 15 novembre 1975.

The coefficient matrices (a_{ij}) and (b_{ij}) are assumed to be symmetric, positive definite, and of class C^1 while p and q are simply continuous in the cylinder

$$\Omega_\tau = \{(x\,,t)\mid x \in D\,,\tau \leq t \leq T\}$$

$\tau \geq 0\,,T < \infty$, where D is a bounded domain in E^n with smooth boundary. By a solution of $Lu = 0$ or $Mv = 0$ we shall mean a function that is twice continuously differentiable in the interior of Ω_τ and continuously differentiable in the closure $\bar{\Omega}_\tau$.

It will be seen that the parameter k plays a role in our results.

2. THE CASE $(a_{ij}) = (b_{ij})\,,i\,,j = 1\,,\cdots,n$

We treat first the case when L and M have the same principal parts, that is, $a_{ij} = b_{ij}$, for $i\,,j = 1\,,\cdots,n$. We consider the boundary value problems

$$(1)\qquad \begin{aligned} &u_{tt} + \frac{k}{t}\,u_t - (a_{ij}\,u_{x_i})_{x_j} + pu = 0 \qquad \text{in}\quad \Omega_0\,,\\ &\frac{\partial u}{\partial n} + r\,(x\,,t)\,u = 0 \qquad \text{on}\quad \partial D\times[0\,,T] \end{aligned}$$

and

$$(2)\qquad \begin{aligned} &v_{tt} + \frac{k}{t}\,v_t - (b_{ij}\,v_{x_i})_{x_j} + qv = 0 \qquad \text{in}\quad \Omega_0\,,\\ &\frac{\partial v}{\partial n} + s\,(x\,,t)\,v = 0 \qquad \text{on}\quad \partial D\times[0\,,T] \end{aligned}$$

where $\partial u/\partial n = a_{ij}\,u_{x_i}\,\nu_j$, $(\nu_1\,,\cdots,\nu_n)$ being the outward unit normal vector on ∂D, and r and s are continuous functions on $\partial D\times[0\,,T]$. When $k > 0$ we have the following result.

THEOREM 1. *Let $k > 0$ and suppose that $p \leq q$ and $r \leq s$. If there exists a solution of the problem* (1) *which is positive for $0 < t < T$ such that $u\,(x\,,t) = 0$, then every solution of the problem* (2) *has a zero in Ω_0.*

Proof. Let v be a solution of the problem (2) and suppose that $v\,(y\,,t) > 0$ in Ω_0. Then by integrating the identity

$$t^k\left[v\left(u_{tt} + \frac{k}{t}\,u_t\right) - u\left(v_{tt} + \frac{k}{t}\,v_t\right)\right] \equiv$$

$$\equiv [t^k\,(u_t\,v - v_t\,u)]_t = t^k\,[a_{ij}\,(u_{x_i}\,v - v_{x_i}\,u)]_{x_j} + t^k\,(q - p)\,uv$$

over Ω_0 and using the divergence theorem, we obtain

$$\int_D [t^k (u_t v - v_t u)]_0^T \, dx = \tag{3}$$

$$= \int_0^T \int_{\partial D} t^k \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) dx \, dt + \iint_{\Omega_0} t^k (q - p) \, uv \, dx \, dt .$$

Substituting the boundary conditions satisfied by u and v, equation (3) yields

$$\int_D T^k u_t (x , T) \, v (x , T) \, dx = \tag{4}$$

$$= \int_0^T \int_{\partial D} t^k (s - r) \, uv \, dx \, dt + \iint_{\Omega_0} t^k (q - p) \, uv \, dx \, dt.$$

From the assumption on p , q , r, and s, it follows that the righthand side of equation (4) is non-negative. But since $u > 0$ for $0 < t < T$ and $u (x , T) = 0$, it follows that $u_t (x , T) < 0$, so that the lefthand side of equation (4) is negative. Thus we have a contradiction and so the theorem is proved.

It is clear from the proof that if $p < q$ or $r < s$, then we can conclude in the theorem that v must have a zero in the interior of Ω_0.

We note that in Theorem 1 no condition on u need be prescribed at the singular line $t = 0$. This is quite in contrast to the case when $k \leq 0$, which includes the normal hyperbolic equation ($k = 0$), as we shall see in the next theorem.

THEOREM 2. *Let $k \leq 0$ and suppose that $p \leq q$ and $r \leq s$. If there exists a solution u of the adjoint of* $Lu = 0$ *that is,*

$$L^* u = u_{tt} - \left(\frac{k}{t} u \right)_t - (a_{ij} u_{x_i})_{x_j} + pu = 0 \qquad \text{in} \quad \Omega_0$$

which is positive for $0 < t < T$ *such that* $u (x , 0) = 0$, $u (x , T) = 0$ *for* $x \in D$ *and* $(\partial u / \partial n) + ru = 0$ *on* $\partial D \times [0 , T]$, *then every solution v of the problem* (2) *has a zero in* Ω_0.

Proof. Again, let v be a solution of the problem (2) and suppose that $v > 0$ in Ω_0. By integrating the identity

$$uMv - vL^* u \equiv \left(uv_t - vu_t + k \frac{uv}{t} \right)_t$$

$$- [a_{ij} (uv_{x_i} - vu_{x_i})]_{x_j} + (q - p) \, uv$$

over Ω_ε and substituting the boundary conditions satisfied by u and v, we obtain

$$\int_D \left\{ u_t(x, T) v(x, T) + \left[uv_t - vu_t + k \frac{uv}{t} \right]_{t=\varepsilon} \right\} dx \tag{5}$$

$$= \int_\varepsilon^T \int_{\partial D} (s - r) uv \, dx \, dt + \iint_{\Omega_\varepsilon} (q - p) uv \, dx \, dt.$$

Now we let ε tend to zero. Since $u(x, 0) = 0$, we note that $\lim_{\varepsilon \to 0} u(x, \varepsilon)/\varepsilon = u_t(x, 0+)$. Thus as $\varepsilon \to 0$, the boundary integral yields

$$\int_D [u_t(x, T) v(x, T) + (k - 1) u_t(x, 0) v(x, 0)] \, dx$$

which is negative since $u_t(x, 0) > 0$, $u_t(x, T) < 0$, and $k \leq 0$. This contradicts the fact that the righthand side in equation (5) is non-negative.

If we consider the problems (1) and (2) in the cylinder Ω_τ where $\tau > 0$, then a combination of the conditions in Theorems 1 and 2 leads to the following result which is valid for any value of the parameter k.

COROLLARY 1. *If there exists a solution of* $Lu = 0$ *in* Ω_τ *which is positive for* $\tau < t < T$ *such that* $u(x, \tau) = 0$, $u(x, T) = 0$, *and* $\partial u/\partial n + ru = 0$ *on* $\partial D \times [\tau, T]$, *then every solution* v *of* $Mv = 0$ *in* Ω_τ *satisfying the condition* $\partial v/\partial n + sv = 0$ *on* $\partial D \times [\tau, T]$ *has a zero in* Ω_τ.

The proof is similar to that of Theorem 1.

3. THE CASE $(a_{ij}) \leq (b_{ij})$

The results obtained above can be extended to the case when $a_{ij} \leq b_{ij}$, $i, j = 1, \cdots, n$, provided the coefficients a_{ij}, p, r, and b_{ij}, q, s are all independent of the variable t, so that L and M are both variable separable.

THEOREM 3. *Let* u *be a nontrivial solution of the problem*

$$\begin{aligned} & u_{tt} + \frac{k}{t} u_t - (a_{ij}(x) u_{x_i})_{x_j} + p(x) u = 0 \qquad in \quad \Omega_0 \\ & \frac{\partial u}{\partial n} + r(x) u = 0 \qquad on \quad \partial D \times [0, T] \end{aligned} \tag{6}$$

such that $u(x, T) = 0$ *for* $k < 0$ *or* $u(x, 0) = 0$ *and* $u(x, T) = 0$ *for* $k \leq 0$, *where* $p \geq 0, r \geq 0$, *and* p, r *are not both identically zero. If*

(7)
$$a_{ij}(x) \leq b_{ij}(x), \qquad i, j = 1, \cdots, n$$
$$p(x) \leq q(x) \quad \text{and} \quad r(x) \leq s(x)$$

where at least one strict inequality holds throughout D, *then every solution* v *of*

(8)
$$v_{tt} + \frac{k}{t} v_t - (b_{ij}(x) v_{x_i})_{x_j} + q(x) v = 0 \quad \text{in} \quad \Omega_0$$
$$\frac{\partial v}{\partial n_b} + s(x) v = 0 \quad \text{on} \quad \partial D \times [0, T]$$

$(\partial v / \partial n_b = b_{ij} v_{x_i} \nu_j)$ *has a zero in the interior of* Ω_0.

Proof. Let φ and ψ be the positive normalized eigenfunctions corresponding to the first eigenvalues λ_0 and μ_0 of the problems

(9)
$$-(a_{ij}(x) w_{x_i})_{x_j} + p(x) w = \lambda w \quad \text{in} \quad D$$
$$\frac{\partial w}{\partial n} + r(x) w = 0 \quad \text{on} \quad \partial D$$

and

(10)
$$-(b_{ij}(x) w_{x_i})_{x_j} + q(x) w = \mu w \quad \text{in} \quad D$$
$$\frac{\partial w}{\partial n_b} + s(x) w = 0 \quad \text{on} \quad \partial D$$

respectively. By using variational principles, we may characterize the eigenvalues λ_0 and μ_0 as

(11)
$$\begin{aligned} 0 < \lambda_0 &= \min_{w \in \Phi} \int_D (a_{ij} w_{x_i} x_{x_j} + p w^2) \, dx + \int_{\partial D} r w^2 \, dx \\ &= \int_D (a_{ij} \varphi_{x_i} \varphi_{x_j} + p \varphi^2) \, dx + \int_{\partial D} r \varphi^2 \, dx \\ &< \int_D (b_{ij} \psi_{x_i} \psi_{x_j} + q \psi^2) \, dx + \int_{\partial D} s \psi^2 \, dx \\ &= \min_{w \in \Psi} \int_D (b_{ij} w_{x_i} w_{x_j} + q w^2) \, dx + \int_{\partial D} s w^2 \, dx = \mu_0 \end{aligned}$$

where Φ and Ψ are the sets of C^2 functions with unit L^2 norm satisfying the boundary conditions of the problems (9) and (10) respectively. Notice that the inequality in equation (11) is due to the assumption made on equation (7).

Now let v be a solution of the problem (8) and let us define

$$U(t) = \int_D u(x, t)\, \varphi(x)\, dx \qquad \text{and} \quad V(t) = \int_D v(x, t)\, \psi(x)\, dx'$$

Then from equations (6) and (9), we have

$$U'' + \frac{k}{t} U' + \lambda_0 U = 0 \tag{12}$$

with the boundary conditions $U(T) = 0$ for $k > 0$, and $U(0) = 0$, $U(T) = 0$ for $k \leq 0$. Likewise from (8) and (10) we have

$$V'' + \frac{k}{t} V' + \mu_0 V = 0. \tag{13}$$

If $k \geq 1$, then a solution of equation (12) which is bounded at $t = 0$ is given by

$$U(t) = t^{(t-k)/2} J_{(k-1)/2}(\sqrt{\lambda_0}\, t)$$

where J_k is the Bessel function of the first kind of order k. The condition $U(T) = 0$ thus implies that $J_{(k-1)/2}(\sqrt{\lambda_0}\, T) = 0$. This means that

$$V(t) = t^{(1-k)/2} J_{(k-1)/2}(\sqrt{\mu_0}\, t)$$

which satisfies equation (13), vanishes at $t = (\lambda_0/\mu_0)^{1/2}\, T < T$. Since $\psi(x) > 0$ we conclude from the definition of V that v has a zero in the interior of Ω_0 when $k \geq 1$.

For $0 < k < 1$, a solution of equation (12) which is bounded at $t = 0$ is given by

$$U(t) = t^{(1-k)/2} [C_1 J_{(k-1)/2}(\sqrt{\lambda_0}\, t) + C_2 J_{(1-k)/2}(\sqrt{\lambda_0}\, t)]$$

where C_1 and C_2 are some constants. Since $U(T) = 0$, the constants C_1, C_2 can be determined, and hence the function

$$V(t) = t^{(1-k)/2} [C_1 J_{(k-1)/2}(\sqrt{\mu_0}\, t) + C_2 J_{(1-k)/2}(\sqrt{\mu_0}\, t)]$$

also vanishes at $t = (\lambda_0/\mu_0)^{1/2}\, T < T$. By the same argument as in the previous case, we again see that v has a zero in the interior of Ω_0.

Finally, the case $k \leq 0$ can also be proved in exactly the same manner provided we note that a solution of equation (12) which vanishes at $t = 0$ is given by

$$U(t) = t^{(1-k)/2} J_{(k-1)/2}(\sqrt{\lambda_0}\, t).$$

4. OSCILLATION THEOREMS

We shall now consider the oscillation phenomenon of solutions of the equation $Lu = 0$ in the semi-infinite cylinder $G = \{(x, t) \mid x \in D, 0 \leq \leq t < \infty\}$. We shall say that $Lu = 0$ is oscillatory if every solution of the equation which vanishes on the lateral surface of G has arbitrarily large zeros in the interior of G. First, we shall establish an oscillation criterion in the case when the matrix (a_{ij}) of L is independent of t.

THEOREM 4. *Let $p(x) \geq 0$ and $r(x) \geq 0$ in D where p and r are not both identically zero. If $p(x) \leq q(x, t)$ in G and $r(x) \leq s(x, t)$ on $\partial D \times [0, \infty)$, then every solution v of*

$$v_{tt} + \frac{k}{t} v_t - (a_{ij}(x) v_{x_i})_{x_j} + q(x, t) v = 0$$

satisfying $\partial v/\partial n + sv = 0$ on $\partial D \times [0, \infty)$ has a zero in $G_\tau = \{(x, t) \mid x \in D, \tau \leq t < \infty\}$ for every τ.

Proof. We shall give the proof only for the case $k > 0$ as the proof for $k \leq 0$ is similar. Let $\varphi > 0$ be the eigenfunction corresponding to the first eigenvalue λ_0 of the problem (9). Then

$$u(x, t) = t^{(1-k)/2} J_{(k-1)/2}(\sqrt{\lambda_0}\, t)\, \varphi(x)$$

is a nontrivial solution of problem (6) which vanishes at the sequence of points $t_1 < t_2 < \cdots < t_n < \cdots$, $t_n = z_n/\sqrt{\lambda_0}\ (n \geq 1)$, where z_n are the zeros of the Bessel function $J_{(k-1)/2}(z)$. The theorem then follows by applying Corollary 1 to each of the domains

$$G_n = \{(x, t) \mid x \in D, t_n \leq t \leq t_{n+1}\}, \quad n \geq 1,$$

More generally, let h be any positive function that is continuous in D with L^2 norm. Let Q be a continuous function satisfying the inequality

$$Q(t) \leq \frac{(Pu, h)}{(u, h)}$$

where

$$Pu = -(a_{ij}(x, t) u_{x_i})_{x_j} + p(x, t) u \quad \text{and} \quad (u, h) = \int_D u(x, t) h(x)\, dx$$

for all functions u that are twice continuously differentiable and positive in G for large t and that vanish on the lateral boundary of G. Then the following two theorems generalize Theorems 6 and 7 and [9], respectively.

THEOREM 5. *The hyperbolic equation*

$$n_{tt} + \frac{k}{t} u_t - (a_{ij}(x, t) u_{x_i})_{x_j} + p(x, t) u = 0$$

is oscillatory in G *if the equation*

$$Z'' + \frac{k}{t} Z' + QZ = 0$$

is oscillatory for $t > 0$.

THEOREM 6. *The hyperbolic equation*

$$u_{tt} + \frac{k}{t} u_x - (a_{ij}(x) u_{x_i})_{x_j} + p(x) u = 0$$

is oscillatory in G *if and only if the elliptic equation*

$$-(a_{ij}(x) u_{x_i})_{x_j} + p(x) u = 0$$

is disconjugate in D.

The proofs are similar to those given in [9] with slight modification.

5. CONCLUDING REMARKS

When $k = 0$ the equation $Lu = 0$ reduces to the hyperbolic equation considered by Travis [9]. Hence our results here include those obtained in [9]. Finally, it should be pointed out that comparison results for the corresponding singular elliptic equation $u_{tt} + (k/t) u_t + (a_{ij} u_{x_i})_{x_j} + pu = 0$ have recently been given by Dunninger and Weinacht [11].

REFERENCES

[1] L. M. KUKS (1962) – *Sturm's theorem and oscillation of solutions of strongly elliptic systems*, « Dokl. Akad. Nauk. SSSR », *142*, 32–35.

[2] C. A. SWANSON (1968) – *Comparison and oscillation theory of linear differential equations*, Academic Press, New York.

[3] C. A. SWANSON – *Strong oscillation of elliptic equations on general domains*, « Canadian Math. Bull. » (to appear).

[4] J. B. DIAZ and J. R. MCLAUGHLIN (1969) – *Sturm separation and comparison theorems for ordinary and partial differential equations*, « Atti Accad. Naz. Lincei, Mem. Cl. Sci. fis. mat. nat. », *9*, 135–194.

[5] K. KREITH and C. TRAVIS (1972) – *Oscillation criteria for selfadjoint elliptic equations*, « Pacific J. Math. », *41*, 743–753.

[6] D. R. DUNNINGER (1969) – *Sturmian theorems for parabolic inequalities*, « Rend. Accad. Sci. fis. mat., Napoli », *36*, 406–410.

[7] C. Y. CHAN and E. C. YOUNG (1973) – *Unboundedness of solutions and comparison theorems for time-dependent quasilinear differential matrix inequalities*, « J. Diff. Eqs. », *14*, 195–201.

[8] K. KREITH (1969) – *Sturmian theorems for hyperbolic equations*, « Proc. Amer. Math. Soc. », *22*, 277–281.

[9] C. TRAVIS – *Comparison and oscillation theorems for hyperbolic equations*, (to appear).

[10] C. KAHANE (1973) – *Oscillation theorems for solutions of hyperbolic equations* « Proc. Amer. Math. Soc. », *41*, 183–188.

[11] D. R. DUNNINGER and R. J. WEINACHT (1971) – *Separation and comparison theorems for classes of singular elliptic inequalities and degenerate elliptic inequalities*, « Applicable Analysis », *1*, 43–55.