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Krishna Bihari Lal, Amir Ali Ansari

On wave solutions of Israel and Trollope's new unified field theory in a $V_2 \times V_2$ space-time

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Equazioni a derivate parziali. — *On wave solutions of Israel and Trollope's new unified field theory in a* $V_2 \times V_2$ *space-time.* Nota di KRISHNA BIHARI LAL e AMIR ALI ANSARI, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Israel e Trollope [1] hanno assegnato due serie di equazioni di campo unificato, ottenute assumendo il tensore g_{ij} simmetrico e Γ^k_{ij} non simmetrico. Nel presente lavoro si considerano tali equazioni in uno spazio–tempo dato da una varietà riemanniana di dimensione quattro e classe due che sia il prodotto $V_2 \times V_2$ di due superficie, e si dimostra che allora l'equazione delle onde ammette soluzioni.

1. INTRODUCTION

The first set of field equations of the new theory [1] is given by

$$(1.1) \qquad a)\ \theta^{ik}_{;\mu^*} = 0, \qquad b)\ {}^*\Gamma_i \equiv {}^*\Gamma^s_{\underset{\vee}{is}} = 0, \qquad c)\ \theta^{\underset{\vee}{ik}}_{,k} = 0,$$
$$d)\ R_{\underline{ik}} = \alpha\, M_{ik}(R_\nu), \qquad e)\ R = 0,$$

where

$$(1.2) \qquad a)\ \theta^{ik} = \sqrt{-g}\,(g^{ik} + \alpha R^{\underset{\vee}{ik}}), \qquad b)\ R_{ik} = \Gamma^s_{is,k} - \Gamma^s_{ik,s} +$$
$$+ \Gamma^s_{tk}\Gamma^t_{is} - \Gamma^s_{ts}\Gamma^t_{ik}, \qquad c)\ M_{ik}(R_\nu) = \frac{1}{4} g_{ij} R_{\underset{\vee}{lm}} R^{\underset{\vee}{lm}} - g^{lm} R_{\underset{\vee}{il}} R_{\underset{\vee}{km}},$$
$$d)\ {}^*\Gamma^s_{ik} = \Gamma^s_{ik} + \frac{2}{3}\delta^s_i \Gamma_k, \qquad (\Gamma_k \equiv \Gamma^l_{\underset{\vee}{kl}})$$

and α is a constant. The fundamental tensor g_{ij} and its inverse g^{ij} defined by $g^{ij} g_{ik} = \delta^j_k$ are used to lower and rise the indices of a tensor. A semi-colon (;) followed by an index with asterisk (*) denotes covariant differentiation with respect to the connections ${}^*\Gamma^i_{jk}$. A bar (—) and a hook (v) below two indices denote the symmetry and skew-symmetry respectively between those indices and a comma denotes partial differentiation.

The second set of field equations of the new theory [1] is given by (1.1) *a*), *b*), *c*) and

$$(1.3) \qquad a)\ R_{\underline{ik}} - (1/4\, b)\, g_{ik} = \alpha\, M_{ik}(R_\nu), \qquad b)\ \theta^{ik} = \sqrt{-g}\,(g^{ik} + \alpha R^{\underset{\vee}{ik}}),$$

where it has been assumed that R is a non-vanishing constant given by the relation $bR = 1$.

Israel and Trollope [1] have found wave solutions of the field equations (1.1) and (1.1), *a*)–*c*), (1.3) in a static spherically symmetric space-time. Lal and Singh [2] have found wave solutions of the same set of equations in a cylin-

(*) Nella seduta del 15 novembre 1975.

drically symmetric space-time. In the present paper we propose to find wave solutions of the same set in a $V_2 \times V_2$ space-time given by

$$ds^2 = -A\,(dx^2 + dy^2) - B\,(dz^2 - dt^2)\,, \tag{1.4}$$

where

$$A = A\,(x\,,y) \quad , \quad B = B\,(z\,,t).$$

PART I. – SOLUTIONS OF FIELD EQUATIONS (1.1)

2. *Solution of equations* (1.1), *a*), *b*), *c*). For solving the field equations (1.1), *a*) a second rank non-symmetric convariant tensor ${}^{+}S_{ik}$ "inverse" to the contravariant tensor density θ^{ik} is defined in [1] by the relations

$${}^{+}S_{il}\,\theta^{kl} = \delta_i^k\,(-\det\,\theta^{ik})^{1/2}\,, \qquad \det\,{}^{+}S_{ik} = \det\,\theta^{ik}. \tag{2.1}$$

Using (2.1) the equation (1.1) *a*) is equivalent to the equation (2.2)

$${}^{+}S_{ik;l^*} = 0. \tag{2.2}$$

In this paper we take the tensor field ${}^{+}S_{ik}$ in the form [3]

$$({}^{+}S_{ik}) = \begin{bmatrix} -A & 0 & \rho & -\rho \\ 0 & -A & \sigma & -\sigma \\ -\rho & -\sigma & -B & 0 \\ \rho & \sigma & 0 & B \end{bmatrix} \qquad (x^i \equiv x\,,y\,,z\,,t)\,, \tag{2.3}$$

where ρ and σ are functions of $x\,,y\,,z-t$. The symmetric parts A, B of ${}^{+}S_{ik}$ correspond to the metric (1.4) taken by Lal and Khare [3] and $\rho\,,\sigma$ are the skew-symmetric parts of ${}^{+}S_{ik}$. Using the tensor field (2.3) Lal and Khare [3] have already solved a set of equations of the type (2.2). The field equations (1.1) *b*) are identically satisfied for $i = 1$ and $i = 2$ while equations for $i = 3$ and $i = 4$ are satisfied under the condition

$$\rho_1 + \sigma_2 = 0, \tag{2.4}$$

where ρ_1 and σ_2 stand for $\partial\rho/\partial x$ and $\partial\sigma/\partial y$ respectively.

Using equation (2.3) in (2.1), we get sixteen simultaneous equations from which on solving, we get sixteen components of θ^{ik} which are given by

$$\theta^{ik} = \begin{bmatrix} -B & 0 & \rho & \rho \\ 0 & -B & \sigma & \sigma \\ -\rho & -\sigma & (\varphi - B)/B & \varphi A/B \\ -\rho & -\sigma & \varphi A/B & (\varphi + B)/B \end{bmatrix}, \tag{2.5}$$

where

$$\varphi = \alpha'/A = (\rho^2 + \sigma^2)/A. \tag{2.6}$$

(The details of calculations have been omitted for the sake of brevity).

Using (2.5) we find that the field equation (1.1), *c*) is identically satisfied if the relation (2.4) holds.

3. *Calculation of* g_{ik}, $\underset{\vee}{R}_{ik}$ *and* $M_{ik}(R_\nu)$. Using (2.5) in equation (1.2), *a*) the non-vanishing components of g^{ik} are given by

$$\begin{aligned} g^{11} = g^{22} &= -1/A \quad , \quad g^{34} = g^{43} = \varphi/B^2 , \\ g^{33} &= (\varphi - B)/B^2 \quad , \quad g^{44} = (\varphi + B)/B^2 ; \end{aligned} \tag{3.1}$$

and the non-vanishing components of $\underset{\vee}{R}^{ik}$ are given by

$$\begin{aligned} \underset{\vee}{R}^{13} = \underset{\vee}{R}^{14} = -\underset{\vee}{R}^{31} = -\underset{\vee}{R}^{41} = \rho/\alpha AB , \\ \underset{\vee}{R}^{23} = \underset{\vee}{R}^{24} = -\underset{\vee}{R}^{32} = -\underset{\vee}{R}^{42} = \sigma/\alpha AB. \end{aligned} \tag{3.2}$$

The contravariant counterparts of g^{ik} and $\underset{\vee}{R}^{ik}$ are given by

$$\begin{aligned} g_{11} = g_{22} = -A \quad , \quad g_{34} = g_{43} = \varphi , \\ g_{33} = -(\varphi + B) \quad , \quad g_{44} = -(\varphi - B) ; \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \underset{\vee}{R}_{12} = \underset{\vee}{R}_{34} = 0 \quad , \quad \underset{\vee}{R}_{13} = -\underset{\vee}{R}_{14} = \rho/\alpha , \\ \underset{\vee}{R}_{23} = -\underset{\vee}{R}_{24} = \sigma/\alpha . \end{aligned} \tag{3.4}$$

Using (3.1)–(3.4) into (1.2), *c*) the components of $M_{ik}(R_\nu)$ are

$$\begin{aligned} M_{11}(R_\nu) &= M_{22}(R_\nu) = 0 , \\ M_{33}(R_\nu) &= M_{44}(R_\nu) = -M_{34}(R_\nu) = \alpha^{-2}(\rho^2 + \sigma^2)/A = \varphi/\alpha^2 . \end{aligned} \tag{3.5}$$

4. *Solutions of equations* (1.1), *d*) *e*). If we denote the Ricci tensor formed from the connections ${}^*\Gamma^i_{jk}$ by ${}^*R_{ik}$, then from equations (1.2), *b*), *d*) we have

$${}^*R_{ik} = R_{ik} + (2/3)(\Gamma_{i,k} - \Gamma_{k,i}) \quad , \quad {}^*R_{\underline{ik}} = R_{\underline{ik}} . \tag{4.1}$$

Substituting the components of ${}^*R_{\underline{ik}}$ from [3] and $M_{ik}(R_\nu)$ from (3.5) in (1.1), *d*), we get

$$L = 0 , \tag{4.2}$$

$$N + 2P = 2\alpha'/\alpha A , \tag{4.3}$$

$$N - 2P = -2\alpha'/\alpha A , \tag{4.4}$$

$$P = \alpha'/\alpha A; \tag{4.5}$$

where

$$L = (A_{11} + A_{22})/A - (A_1^2 + A_2^2)/A^2 ,$$

$$N = (B_{33} - B_{44})/B - (B_3^2 - B_4^2)/B^2 ,$$

$$P = 3\alpha' (A_1^2 + A_2^2)/2A^4 - \alpha' (A_{11} + A_{22})/A^3 + $$
$$+ (\alpha'_{11} + \alpha'_{22})/2A^2 - (\alpha'_1 A_1 + \alpha'_2 A_2)/A^3 - (\rho_2 - \sigma_1)^2/2A^2.$$

By virtue of (4.4) and (2.6), (4.5) can be written in the form

$$\varphi_{11} + \varphi_{22} + \varphi_{33} - \varphi_{44} = 4A\{\beta'^2 + (\varphi/2\alpha)\} . \tag{4.6}$$

where $$\beta' = (\rho_2 - \sigma_1)/2A.$$

Using equation (4.5) in (4.3) and (4.4), we get

$$N = 0 . \tag{4.7}$$

Putting

$$a' = \log A \qquad \text{and} \qquad b' = \log B \tag{4.8}$$

in equations (4.2) and (4.7), we get

$$a'_{11} + a'_{22} = 0 , \tag{4.9}$$

$$b'_{33} - b'_{44} = 0 . \tag{4.10}$$

Equation (4.9) is a Laplacian in a' and has a solution of the form

$$a' = \log A = f_1(x + iy) + f_2(x - iy) , \tag{4.11}$$

while (4.10) is a wave equation and has a solution of the form

$$b' = \log B = g_1(z + t) + g_2(z - t) , \tag{4.12}$$

where f_1, f_2, g_1 and g_2 are functions of their arguments.

Hence the solution of field equation (1.1), *d*) is given by (4.11), (4.12) and (4.6).

The field equation (1.1), *e*) is identically satisfied since, by using (3.1), (4.1) and ${}^*R_{\underline{ij}}$ from [3] in $R = g^{ij} R_{ij}$, we get

$$R = g^{ij} R_{ij} = g^{ij} R_{\underline{ij}} = g^{ij}\, {}^*R_{\underline{ij}} = -(L/A + N/B)$$

which is identically equal to zero in view of (4.2) and (4.7).

PART 2. SOLUTIONS OF FIELD EQUATIONS (1.2), *a*) – *c*) AND (1.4).

The field equations (1.1), *a*)–*c*) have already been solved in Section 2 and the field equation (1.3), *b*) is identical with (1.2), *a*), which has been solved in section 3. We have now to find only the solution of (1.3), *a*).

5. *Solution of equation* (1.3), *a*). Using the components of $^*R_{\underline{ij}}$ from [3], (3.3), (3.5) in equation (1.3), *a*), we get

$$L = -A/2b. \tag{5.1}$$

$$N + 2P = (2\alpha'/\alpha A) - (\varphi + B)/2b, \tag{5.2}$$

$$N - 2P = -(2\alpha'/\alpha A) + (\varphi - B)/2b, \tag{5.3}$$

$$P = (\alpha'/\alpha A) - (\varphi/4b). \tag{5.4}$$

Equation (5.4) can be written in a slightly modified form as

$$\varphi_{11} + \varphi_{22} + \varphi_{33} - \varphi_{44} = 4A\{\beta'^2 + (\varphi/2\alpha) - (\varphi/8b)\}. \tag{5.5}$$

Using (5.4) in (5.2) and (5.3), we get

$$N = -B/2b. \tag{5.6}$$

Using (4.8) in (5.1) and (5.6), we get

$$a'_{11} + a'_{22} = -e^{a'}/2b, \tag{5.7}$$

and

$$b'_{33} - b'_{44} = -e^{b'}/2b. \tag{5.8}$$

Reducing equation (5.7) to canonical form by changing the dependent variable a' into ζ where $\zeta = \zeta(\xi, \eta)$ and $\xi = x + iy$ and $\eta = x - iy$, we get

$$\partial^2\zeta/\partial\xi\,\partial\eta = -(1/b)\,e^{\zeta}. \tag{5.9}$$

Equation (5.9) is of Liouville's form and, by Forsyth [4], has a solution of the form

$$e^{\zeta} = 2\bar{f}_1(\xi)\bar{f}_2(\eta)/[f_1(\xi) - (1/8b)f_2(\eta)]^2. \tag{5.10}$$

The exact solution of (5.7) is

$$\begin{aligned} a' = \log A = \log[2\bar{f}_1(x+iy)] + \log[\bar{f}_2(x-iy)] - \\ - 2\log[f_1(x+iy) - (1/8b)\{f_2(x-iy)\}], \end{aligned} \tag{5.11}$$

Similarly, an exact solution of (5.8) is

$$b' = \log \mathrm{B} = \log [2 \bar{g}_1 (z + t)] + \log [\bar{g}_2 (z - t)] - \\ - 2 \log [g (z + t) - (1/8 b) \{g_2 (z - t)\}] , \tag{5.12}$$

where f_1, f_2, g_1 and g_2 are arbitrary functions of their arguments and bars denote the partial differentations with respect to them. Hence the solution of field equation (1.3), *a*) is given by (5.5), (5.11) and (5.12).

References

[1] W. Israel and R. Trollope (1961) – *New possibilities of a unified field theory*, «Math. Phys. (U.S.A.)», *2*, 777–786.

[2] K. B. Lal and T. Singh (1973) – *On cylindrical wave solutions of Israel and Trollope's new unified field theory*, «Tensor», *27*, 161–169.

[3] K. B. Lal and D. C. Khare (1969) – *On solutions of Eintein's unified field equations of gravitation and electromagnetism in a $V_2 \times V_2$ space–time*, «Tensor (N.S.)», *20*, 335–342.

[4] A. R. Forsyth (1953) – *A treatise on Differential Equations*, «McMillan and Co. Ltd.», p. 555.