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On wave solutions of Israel and Trollope’s new unified field theory in a $V_2 \times V_2$ space-time


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Equazioni a derivate parziali. — On wave solutions of Israel and Trollope's new unified field theory in a $V_2 \times V_2$ space-time. Nota di Krishna Bihari Lal e Amir Ali Ansari, presentata (1) dal Socio B. Segre.

Riassunto. — Israel e Trollope [1] hanno assegnato due serie di equazioni di campo unificato, ottenute assumendo il tensore $g_{ij}$ simmetrico e $\Gamma^k$ non simmetrico. Nel presente lavoro si considerano tali equazioni in uno spazio–tempo dato da una varietà riemanniana di dimensione quattro e classe due che sia il prodotto $V_2 \times V_2$ di due superficie, e si dimostra che allora l'equazione delle onde ammette soluzioni.

I. INTRODUCTION

The first set of field equations of the new theory [1] is given by

\[
\begin{align*}
(1.1) \quad & a) \quad \theta_{ik} = 0, \\
& b) \quad \ast \Gamma_i = \ast \Gamma^i = 0, \\
& c) \quad \theta_{ik} = 0, \\
& d) \quad R_{ik} = \alpha M_{ik} (R_v), \\
& e) \quad R = 0,
\end{align*}
\]

where

\[
\begin{align*}
(1.2) \quad & a) \quad \theta_{ik} = \sqrt{-g} (g^{ik} + \alpha R_{ik}), \\
& b) \quad R_{ik} = \Gamma_{is,k} - \Gamma_{ik,s} + \\
& \quad + \Gamma_{ik} \Gamma^s_{is} - \Gamma^s_{is} \Gamma_{ik}, \\
& c) \quad M_{ik} (R_v) = \frac{1}{4} g_{ij} R_{lm} R_{lm} - g_{lm} R_{il} R_{km}, \\
& d) \quad \ast \Gamma^i = \Gamma^i + \frac{2}{3} \delta^i \Gamma_k, \\
& (\Gamma_k = \Gamma^i_{ik})
\end{align*}
\]

and $\alpha$ is a constant. The fundamental tensor $g_{ij}$ and its inverse $g^{ij}$ defined by $g^{ij} g_{ik} = \delta^i_k$ are used to lower and rise the indices of a tensor. A semi-colon (;) followed by an index with an asterisk (*) denotes covariant differentiation with respect to the connections $\ast \Gamma^i_{jk}$. A bar (—) and a hook (v) below two indices denote the symmetry and skew-symmetry respectively between those indices and a comma denotes partial differentiation.

The second set of field equations of the new theory [1] is given by (1.1) $a), b), c)$ and

\[
(1.3) \quad a) \quad R_{ik} = (1/4 \beta) g_{ik} = \alpha M_{ik} (R_v), \\
\]

where it has been assumed that $R$ is a non-vanishing constant given by the relation $\beta R = 1$.

Israel and Trollope [1] have found wave solutions of the field equations (1.1) and (1.1), $a)-c)$, (1.3) in a static spherically symmetric space-time. Lal and Singh [2] have found wave solutions of the same set of equations in a cylin-

(*) Nella seduta del 15 novembre 1975.
drically symmetric space-time. In the present paper we propose to find wave solutions of the same set in a $V_2 \times V_2$ space-time given by

$$ds^2 = -A(dx^2 + dy^2) - B(ds^2 - dt^2),$$

where

$$A = A(x, y), \quad B = B(z, t).$$

**Part I. Solutions of field equations (1.1)**

2. Solution of equations (1.1), a), b), c). For solving the field equations (1.1), a) a second rank non-symmetric contravariant tensor $\pm S^i_{jk}$ is defined in [1] by the relations

$$(2.1) \quad \pm S^i_{jk} = \delta^i_k \left(-\det \theta^{ik}\right)^{1/2}, \quad \det \pm S_{ik} = \det \theta^{ik}.$$  

Using (2.1) the equation (1.1) a) is equivalent to the equation (2.2)

$$(2.2) \quad \pm S_{ik}^{i*} = 0.$$  

In this paper we take the tensor field $\pm S_{ik}$ in the form [3]

$$(2.3) \quad (\pm S_{ik}) = \begin{pmatrix} -A & \rho & -\rho \\ 0 & -A & \sigma \\ -\rho & -\sigma & -B \\ \rho & \sigma & o & B \end{pmatrix}, \quad (x^i = x, y, z, t),$$

where $\rho$ and $\sigma$ are functions of $x, y, z, t$. The symmetric parts $A, B$ of $\pm S_{ik}$ correspond to the metric (1.4) taken by Lal and Khare [3] and $\rho, \sigma$ are the skew-symmetric parts of $\pm S_{ik}$. Using the tensor field (2.3) Lal and Khare [3] have already solved a set of equations of the type (2.2). The field equations (1.1) b) are identically satisfied for $i = 1$ and $i = 2$ while equations for $i = 3$ and $i = 4$ are satisfied under the condition

$$(2.4) \quad \rho_1 + \sigma_2 = 0,$$

where $\rho_1$ and $\sigma_2$ stand for $\partial \rho/\partial x$ and $\partial \sigma/\partial y$ respectively.

Using equation (2.3) in (2.1), we get sixteen simultaneous equations from which on solving, we get sixteen components of $\theta^{ik}$ which are given by

$$(2.5) \quad \theta^{ik} = \begin{pmatrix} -B & \rho & \rho \\ o & -B & \sigma \\ -\rho & -\sigma & (\phi - B)/B \quad \phi A/B \\ -\rho & -\sigma & \phi A/B \quad (\phi + B)/B \end{pmatrix},$$
where

\[ \varphi = \frac{x'}{A} = \frac{\rho^2 + \sigma^2}{A}. \]

(The details of calculations have been omitted for the sake of brevity).

Using (2.5) we find that the field equation (1.1), \( e \) is identically satisfied if the relation (2.4) holds.

3. Calculation of \( g_{ik} \), \( R_{ik} \) and \( M_{ik} \). Using (2.5) in equation (1.2), 
\( a \) the non-vanishing components of \( g_{ik} \) are given by

\[ g^{11} = g^{22} = -\frac{1}{A} , \quad g^{34} = g^{43} = \frac{\varphi}{B^2}, \]
\[ g^{33} = (\varphi - B)/B^2 , \quad g^{44} = (\varphi + B)/B^2 ; \]

and the non-vanishing components of \( R_{ik} \) are given by

\[ R^{12} = R^{14} = -R^{24} = -R^{41} = \rho/\alpha AB, \]
\[ R^{23} = R^{34} = -R^{32} = -R^{43} = \sigma/\alpha AB. \]

The contravariant counterparts of \( g_{ik} \) and \( R_{ik} \) are given by

\[ g^{11} = g^{22} = -A , \quad g^{34} = g^{43} = \varphi, \]
\[ g^{33} = -(\varphi + B), \quad g^{44} = -(\varphi - B); \]

and

\[ R_{12} = R_{34} = 0 , \quad R_{13} = -R_{14} = \rho/\alpha, \]
\[ R_{23} = -R_{34} = \sigma/\alpha. \]

Using (3.1)-(3.4) into (1.2), \( c \) the components of \( M_{ik} (R_v) \) are

\[ M_{11} (R_v) = M_{22} (R_v) = 0 , \]
\[ M_{33} (R_v) = M_{44} (R_v) = -M_{34} (R_v) = \alpha^{-2} (\rho^2 + \sigma^2)/A = \varphi/\alpha^2 . \]

4. Solutions of equations (1.1), \( d \). If we denote the Ricci tensor formed from the connections \( ^*\Gamma_{ik}^j \) by \( ^*R_{ik} \), then from equations (1.2), \( b \), \( d \) we have

\[ ^*R_{ik} = R_{ik} + (2/3) (\Gamma_{i,k}^j - \Gamma_{j,k}^i) , \quad ^*R_{ik} = R_{ik}. \]

Substituting the components of \( ^*R_{ik} \) from [3] and \( M_{ik} (R_v) \) from (3.5) in (1.1), \( d \), we get

\[ L = 0 , \]
\[ N + 2P = 2x'/\alpha A , \]
\[ N - 2P = -2x'/\alpha A , \]
\[ P = x'/\alpha A; \]

where
\[ L = (A_{11} + A_{22})/A - (A_1^2 + A_2^2)/A^2, \]
\[ N = (B_{33} - B_{44})/B - (B_3^2 - B_4^2)/B^2, \]
\[ P = 3 \alpha'(A_1^2 + A_2^2)/2 A^4 - \alpha'(A_{11} + A_{22})/A^3 + \]
\[ + (\alpha'_{11} + \alpha'_{22})/2 A^2 - (\alpha' A_1 + \alpha' A_2)/A^3 - (\rho_2 - \sigma_1)^2/2 A^2. \]

By virtue of (4.4) and (2.6), (4.5) can be written in the form
\[ (4.6) \varphi_{11} + \varphi_{22} + \varphi_{33} - \varphi_{44} = 4A \{ \beta'^2 + (\varphi/2 \alpha) \}. \]
where
\[ \beta' = (\rho_2 - \sigma_1)/2 A. \]

Using equation (4.5) in (4.3) and (4.4), we get
\[ (4.7) N = 0. \]

Putting
\[ (4.8) a' = \log A \quad \text{and} \quad b' = \log B \]
in equations (4.2) and (4.7), we get
\[ (4.9) a'_{11} + a'_{22} = 0, \]
\[ (4.10) b'_{33} - b'_{44} = 0. \]

Equation (4.9) is a Laplacian in \(a'\) and has a solution of the form
\[ (4.11) a' = \log A = f_1(x + iy) + f_2(x - iy), \]
while (4.10) is a wave equation and has a solution of the form
\[ (4.12) b' = \log B = g_1(s + t) + g_2(s - t), \]
where \(f_1, f_2, g_1\) and \(g_2\) are functions of their arguments.

Hence the solution of field equation (1.1), \(e\) is given by (4.11), (4.12) and (4.6).

The field equation (1.1), \(e\) is identically satisfied since, by using (3.1), (4.1) and \(^*R_{ij}\) from [3] in \(R = g^{ij} R_{ij}\), we get
\[ R = g^{ij} R_{ij} = g^{ij} R_{ij} = g^{ij} *R_{ij} = -(L/A + N/B) \]
which is identically equal to zero in view of (4.2) and (4.7).
PART 2. SOLUTIONS OF FIELD EQUATIONS \((1.2), a) - c) AND \((1.4)\).

The field equations \((1.1), a) - c)\) have already been solved in Section 2 and the field equation \((1.3), b)\) is identical with \((1.2), a)\), which has been solved in section 3. We have now to find only the solution of \((1.3), a)\).

5. Solution of equation \((1.3), a)\). Using the components of \(\mathbf{R}_{ij}\) from \([3], (3.3), (3.5)\) in equation \((1.3), a)\), we get

\[
\begin{align*}
L &= -A/2b, \\
N + 2P &= (2a'/xA) - (\varphi + B)/2b, \\
N - 2P &= - (2a'/xA) + (\varphi - B)/2b, \\
P &= (a'/xA) - (\varphi/4b).
\end{align*}
\]

Equation \((5.4)\) can be written in a slightly modified form as

\[
\varphi_{11} + \varphi_{22} + \varphi_{33} - \varphi_{44} = 4A \left\{ \beta^2 + (\varphi/2a) - (\varphi/8b) \right\}.
\]

Using \((5.4)\) in \((5.2)\) and \((5.3)\), we get

\[
N = -B/2b.
\]

Using \((4.8)\) in \((5.1)\) and \((5.6)\), we get

\[
\begin{align*}
a_{11}' + a_{22}' &= -\varepsilon a'/2b, \\
\text{and}
\end{align*}
\]

\[
\begin{align*}
b_{22}' - b_{44}' &= -\varepsilon b'/2b.
\end{align*}
\]

Reducing equation \((5.7)\) to canonical form by changing the dependent variable \(a'\) into \(\zeta\) where \(\zeta = \zeta(\xi, \eta)\) and \(\xi = x + iy\) and \(\eta = x - iy\), we get

\[
\partial^2 \zeta / \partial \xi^2 \partial \eta = -(1/b) \varepsilon^5.
\]

Equation \((5.9)\) is of Liouville's form and, by Forsyth \([4]\), has a solution of the form

\[
\varepsilon^5 = 2f_1(\xi) f_2(\eta) / [f_1(\xi) - (1/8b) f_2(\eta)]^2.
\]

The exact solution of \((5.7)\) is

\[
a' = \log A = \log \left[2f_1(x + iy) + \log \left[f_2(x - iy)\right]\right] - 2 \log \left[f_1(x + iy) - (1/8b) \left\{ f_2(x - iy)\right\}\right],
\]

\[
\text{Subject:}
\text{The exact solution of (5.7) is}
\text{a' = log A = log [2f_1(x + iy) + log [f_2(x - iy)]] - 2 log [f_1(x + iy) - (1/8b) {f_2(x - iy)}],}
\]
Similarly, an exact solution of (5.8) is

\begin{equation}
\begin{aligned}
 b' &= \log B = \log \left[ 2 \tilde{g}_1 (\varepsilon + \tau) \right] + \log \left[ \tilde{g}_2 (\varepsilon - \tau) \right] - \\
 & \quad - 2 \log \left[ g (\varepsilon + \tau) - (1/8b) \{ g_1 (\varepsilon + \tau) \} \right],
\end{aligned}
\end{equation}

where \( f_1, f_2, g_1 \) and \( g_2 \) are arbitrary functions of their arguments and bars denote the partial differentiations with respect to them. Hence the solution of field equation (1.3), a) is given by (5.1), (5.11) and (5.12).

References