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Some measure theoretic properties of completely regular spaces. Nota I

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Analisi matematica. — *Some measure theoretic properties of completely regular spaces.* Nota I di A. G. A. G. BABIKER, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — In questa Nota I ed in una successiva Nota II gli spazi completamente regolari compatti rispetto ad una misura vengono studiati col nome di spazi « essentially Lindelöf », ottenendone due diverse caratterizzazioni.

1. INTRODUCTION

The study of certain classes of completely regular spaces which are characterized by their measure theoretic properties have received the attention of a few authors in the literature, e.g. [8], [10], [11] and [14]. One such class is defined by the requirement that every Baire measure be net additive. We call these spaces *essentially Lindelöf* spaces (the term " measure compact " has also been used [11]).

We give two characterizations of these spaces. In § 2 the notion of a sequential subspace of $C^*(X)$, the ring of the bounded continuous functions on X, is introduced and is used to characterize essentially Lindelöf spaces as well as realcompact and pseudocompact spaces. In § 3 (Part II of this Note) a locally convex topology σ on $C^*(X)$ is defined, and essentially Lindelöf spaces are characterized in terms of ideals of $C^*(X)$ which are closed w.r.t. σ. This topology which was introduced, in a different form, in [1] and [3], is equivalent to topologies studied extensively in [6] and [13]. However, the description we give here is more suitable to our purposes. Specializing to locally compact spaces in § 4, we give a simple characterization of essentially Lindelöf locally compact spaces, and we show by an example how a locally compact realcompact locally metrizable space can admit measures which are not net additive, thus answering, in the negative, a question raised by Kirk in [9].

The notation we use is, by now, fairly standard. A signed Baire measure on X is a finite realvalued σ-additive set function defined on the Baire sets (the σ-algebra generated by the family $\{f^{-1}(0) : f \in C^*(X)\}$ of zero sets); a Baire measure is a positive valued signed Baire measure. A Baire measure μ is *net-additive* if for each net $\{Z_\alpha\}_{\alpha \in A}$ of zero sets, decreasing to $\varnothing$ (i.e. $Z_\alpha \subset Z_\beta$ if $\beta < \alpha$ and $\bigcap_{\alpha \in A} Z_\alpha = \varnothing$) we have $\mu(Z_\alpha) \to 0$; and μ is *compact-regular* if, given $\varepsilon > 0$, there exists a compact set $K \subset X$ such that, if Z is zero set with $K \cap Z = \varnothing$, then $\mu(Z) < \varepsilon$. This terminology will be applied to signed mea-

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sures if it applies to both the positive and negative parts in the Hahn-Jordan decomposition, and may be transferred to norm bounded linear functionals on $C^*(X)$ by means of the integral representation theorem. For further information we refer to Varadarajan [14] and Knowles [10].

A space in which every Baire measure is compact-regular will be called *essentially compact*. The term strongly measure compact was been used in this context [12]. The use of the term *essentially Lindelöf* for spaces in which every Baire measure is net-additive is justified by the following proposition whose proof is straightforward.

PROPOSITION. X *is essentially Lindelöf if, and only if, for every Baire measure* μ *on* X *and every open cover* $\{G_\alpha\}_{\alpha \in A}$ *of* X, $\exists$ *a countable subcover* $\{G_{\alpha_i}\}$ $(i = 1, 2, \cdots)$ *such that*

$$\mu^*\left(X \setminus \left[\bigcup_{i=1}^{\infty} G_{\alpha_i}\right]\right) = 0 .$$

Throughout, $C^*(X)$ will be denoted by C^* when no ambiguity can arise. By a " subspace " we mean a proper linear subspace of C^* considered as a vector space, and by an " ideal ", we mean a proper ideal of C^* considered as a ring. When no topology on C^* is explicitly given, all topological notions are taken relative to the uniform norm topology.

§ 2. FIRST CHARACTERIZATION OF ESSENTIALLY LINDELÖF SPACE

A family $\mathcal{F} \subset C^*$ will be called *sequential* if for any sequence $\{f_n\}$ in C^* with $f_n \searrow 0$, and any $\varepsilon > 0$, $\exists$ an integer N such that for each integer $m > N$ there exists $g_m \in \mathcal{F}$ satisfying $\|g_m - f_m\| < \varepsilon$. Alternatively, $\mathcal{F}$ is sequential if and only if for any sequence $\{f_n\}$ in C^* with $f_n \searrow 0$ there exists a sequence $\{g_n\} \subset \mathcal{F}$ such that $\|f_n - g_n\| \to 0$.

By a hyperplane in C^*, we mean a uniformly closed vector subspace H of C^* of codimension one, i.e. $C^*/H \cong \mathbf{R}$. Clearly any hyperplane H can be written in the form $H = L^{-1}(0)$, for some linear functional L on C^* with $\|L\| = 1$.

LEMMA 2.1. *A hyperplane* $H = L^{-1}(0)$ *is sequential if and only if* L *is* σ*-additive.*

Proof. Suppose that $H = L^{-1}(0)$ where L is a σ-additive linear functional with $\|L\| = 1$. $\exists g \in C^* \setminus H$ such that for each $f \in C^*$, $\exists h \in H$ satisfying $f = h + L(f)\, g$. Let $f_n \searrow 0$ and $\varepsilon > 0$ be given. Since L is σ-additive, $\exists$N such that for all $m > N$, $|L(f_m)| < \varepsilon/\|g\|$. Write:

$$f_m = h_m + L(f_m)\, g \quad , \quad h_m \in H .$$

Then

$$\|h_m - f_m\| = \|L(f_m)\, g\| = |L(f_m)| \cdot \|g\| < \varepsilon .$$

Thus H is sequential.

Conversely, suppose that H is sequential. Let $f_n \searrow 0$ and $\varepsilon > 0$ be given. By hypothesis, $\exists N$ such that for each $m > N$, $\exists h_m \in H$ such that $\|h_m - f_m\| < \varepsilon$. But, $|L(f_m)| = |L(h_m - f_m)| < \|h_m - f_m\| < \varepsilon$. Therefore $L(f_n) \to 0$, i.e. L is σ-additive.

We now give a characterization of essentially Lindelöf spaces. For the definition of 'fixed' and 'free' ideals, we refer to [7].

THEOREM 2.2. *A completely regular space* X *is essentially Lindelöf if and only if every ideal contained in a closed sequential subspace of* $C^*(X)$ *is fixed.*

Proof. Since every closed subspace is contained in a hyperplane, and since any hyperplane containing a sequential subspace is itself sequential, it is sufficient to prove the theorem with 'subspace' replaced by 'hyperplane'.

Sufficiency. Suppose that no free ideal is contained in a sequential hyperplane. If X is not essentially Lindelöf, X admits a Baire measure μ without support [11]. Let L be the positive functional corresponding to μ. Let $\tilde{\mu}$ be the induced measure on βX, the Stone-Cech compactification of X [10]. For each $f \in C^*$ let $\tilde{f}$ be its unique extension to βX.

Let $K \subset \beta X \setminus X$ be the support of μ. Then $I = \{f \in C^* : \tilde{f}(K) = 0\}$ is a free ideal in C^*.

For each $f \in I$, we have

$$L(f) = \tilde{L}(\tilde{f}) = \int_{\beta X} \tilde{f}\, d\tilde{\mu} = \int_{K} \tilde{f}\, d\tilde{\mu} = 0 .$$

Write $H = L^{-1}(0)$. Then by (2.1) H is a sequential hyperplane. But $I \subset H$. This contradicts the hypothesis.

Necessity. Suppose that X is essentially Lindelöf, and H is a sequential hyperplane in C^* containing a free ideal I. By Lemma (2.1) $H = L^{-1}(0)$ for some σ-additive linear functional L. Let μ be the signed measure corresponding to L. Then $\mu = \mu^+ - \mu^-$, where μ^+, μ^- are Baire measures on X.

Write $K = \bigcap_{f \in I} \tilde{f}^{-1}(0)$. Since I is free $K \subset \beta X \setminus X$. We now show that $\operatorname{supp}(|\tilde{\mu}|) \subset K$, $(|\tilde{\mu}| = \tilde{\mu}^+ + \tilde{\mu}^-)$.

Let $x \in \beta X \setminus K$. Then there exists $f \in I$ such that $\tilde{f}(x) = \alpha > 0$, and $\|f\| = 1$. Write:

$$V = \left\{ y \in \beta X : \tilde{f}(y) > \frac{\alpha}{2} \right\} .$$

Then V is a neighbourhood of x and $V \cap K = \varnothing$.

Let $\beta X = P \cup N$ be a Hahn decomposition of βX with respect to $\tilde{\mu}$. Find compact G_δ sets $Z_1 \subset P \cap V$ and $Z_2 \subset N \cap V$ such that, for a given $\varepsilon > 0$,

$$\tilde{\mu}^+(Z_1) \geq \frac{\tilde{\mu}^+(V)}{2} \quad ; \quad \tilde{\mu}^-(Z_2) \geq \tilde{\mu}^-(V) - \varepsilon .$$

Let $g \in C^*(X)$ be such that,

$$\tilde{g}(Z_1) = 1 \quad ; \quad \tilde{g}(Z_2) = 0 \quad ; \quad \tilde{g}(\beta X \setminus V) = 0; \qquad 0 \leq g \leq 1.$$

Write $h = fg$. Then $h \in I$, and,

$$1 \geq \tilde{h}(Z_1) > \frac{\alpha}{2} \quad ; \quad \tilde{h}(Z_2) = 0;$$

$$\tilde{h}(\beta X \setminus V) = 0 \quad ; \quad 0 \leq h \leq 1 .$$

Now

$$L(h) = \int_X h \, d\mu^+ - \int_X h d\mu^- = \int_{V \setminus Z_2} \tilde{h} \, d\tilde{\mu}^+ - \int_{V \setminus Z_2} \tilde{h} \, d\tilde{\mu}^- \geq$$

$$\geq \int_{Z_1} \tilde{h} \, d\tilde{\mu}^+ - \int_{V \setminus Z_2} \tilde{h} \, d\tilde{\mu}^- \geq \frac{\alpha}{2} \tilde{\mu}^+(Z_1) - \tilde{\mu}^-(V \setminus Z_2) \geq \alpha \frac{\tilde{\mu}^+(V)}{4} - \varepsilon .$$

But $h \in I$. Therefore $L(h) = 0$. Hence $\tilde{\mu}^+(V) = 0$. Similarly $\tilde{\mu}^-(V) = 0$. So $|\tilde{\mu}|(V) = 0$, i.e. $x \notin \operatorname{supp}(|\tilde{\mu}|)$. Thus $|\mu|$ is a Baire measure without support in X. This contradicts the hypothesis that X is essentially Lindelöf and establishes the theorem.

Theorem 4.2 implies that essentially Lindelöf spaces have the property that every sequential ideal in C^* is fixed. The following theorem shows that this property characterizes realcompact spaces.

THEOREM 2.3. *The following conditions are equivalent*:

(i) X *is realcompact*.

(ii) *For every sequential ideal* $I \subset C^*(X)$, $\bigcap_{f \in I} \tilde{f}^{-1}(0) \subset X$.

(where $\tilde{f}$ is the extension of f to βX)

(iii) *Every sequential ideal in* $C^*(X)$ *is fixed*.

(iv) *Every sequential maximal ideal in* $C^*(X)$ *is fixed*.

Proof. We only need to prove (i) ⇒ (ii) and (iv) ⇒ (i).

(i) ⇒ *(ii)*: Suppose that X is realcompact and let $I \subset C^*(X)$ be a sequential ideal. Suppose that $p \in \bigcap_{f \in I} \tilde{f}^{-1}(0)$. Define L_p on C^* by:

$$L_p(f) = \tilde{f}(p).$$

L_p is a non-negative linear functional. Furthermore, $H = L_p^{-1}(0) = \{f \in C^* : \tilde{f}(p) = 0\}$ is a maximal ideal and hence is also a hyperplane. Since H contains the sequential ideal I, H is sequential.

By Lemma 2. L_p is σ-additive. The Baire measure μ corresponding to L_p is the unit-point-mass at p. As X is realcompact, $p \in X$ [2, th. 5.1]. Thus $\bigcap_{f \in I} \tilde{f}^{-1}(0) \subset X$.

(*iv*) $\Rightarrow$ (*i*): Let $p \in \nu X$, the realcompactification of X [cf. 7]. The linear functional L_p, defined as above, is σ-additive. $L_p^{-1}(0)$ is a maximal ideal which is sequential (2.1). By (iv) $L_p^{-1}(0)$ is fixed. i.e. $p \in X$. Thus $X = \nu X$. This completes the proof.

For pseudocompact spaces we have the following characterization.

THEOREM 2.4. *The following statements are equivalent*:

(i) X *is pseudocompact.*

(ii) *Every ideal in* C^* *is contained in a sequential hyperplane.*

(iii) *Every ideal in* C^* *is contained in a closed sequential subspace.*

(iv) *Every maximal ideal in* C^* *is sequential.*

Proof. (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) is plain.

(*i*) $\Rightarrow$ (*ii*): Let I be an ideal in $C^*(X)$ and let $p \in \beta X$ be such that $\tilde{f}(p) = 0$ for all $f \in I$. Let L_p be as in the proof of (2.3). Then $L_p^{-1}(0)$ is a hyperplane containing I. Since X is pseudocompact, L_p is σ-additive [10, th. 3.1]. By (2.1) $L_p^{-1}(0)$ is sequential.

(*iv*) $\Rightarrow$ (*i*): Suppose X is not pseudocompact. Then $\exists$ a non-empty zero set Z in βX such that $Z \cap X = \varnothing$. Let $p \in Z$. Then $L_p^{-1}(0)$ is a maximal ideal. By (iv) $L_p^{-1}(0)$ is sequential. The set function μ corresponding to L_p is induced by the unit-point-mass at p. As $p \in Z \subset \beta X \setminus X$, μ is not σ-additive. This contradicts Lemma (2.1) and completes the proof.

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