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Some invariants for rank three torsion-free modules over a Dedekind domain

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Algebra. — *Some invariants for rank three torsion-free modules over a Dedekind domain.* Nota di LUCIE DE MUNTER-KUYL, presentata (*) dal Corrisp. G. ZAPPA.

RIASSUNTO. — Viene associato un sistema completo di invarianti ad un modulo M di rango tre libero da torsione sopra un dominio di Dedekind e ad una terna di elementi indipendenti di M. I metodi usati sono simili a quelli della teoria dei gruppi abeliani.

1. INTRODUCTION

In [3], we have associated a complete system of invariants with the triple (M, x_1, x_2) consisting of a rank two torsion-free module M and two independent elements of M. The purpose of this paper is to extend our results to modules of rank three.

Let A be a Dedekind domain, K its field of fractions, $\mathcal{P}$ the set of non-zero prime ideals of A, $A_{\mathfrak{p}}$ the local ring of A at the non-zero prime ideal $\mathfrak{p}$, and π a uniformizing element of A.

An (integral) superdivisor of A is defined to be a mapping μ from $\mathcal{P}$ to $\overline{\mathbf{N}} = \mathbf{N} \cup \{0, \infty\}$. Multiplication of superdivisors is defined by $(\mu\mu')(\mathfrak{p}) = \mu(\mathfrak{p}) + \mu'(\mathfrak{p})$, with the convention that $n + \infty = \infty$, $\forall n \in \overline{\mathbf{N}}$. Integral ideals of A are identified with the superdivisor corresponding to their prime decomposition and multiplicative terminology is carried over from ideals to superdivisors. In particular, we write $\mu \mid \mu'$, when μ divides μ', and we denote by $[\mu, \mu']$ the GCD of two arbitrary superdivisors μ and μ'.

In accordance with the group theoretical terminology, we define a torsion A-module T to be $\mathfrak{p}$-primary if every element of T has order a power of $\mathfrak{p}$, i.e. if the submodule zero is $\mathfrak{p}$-primary in T, in the usual sense.

For $k \in \mathbf{N} \cup \{0\}$, we denote by $A(\mathfrak{p}^k)$ the $\mathfrak{p}$-primary A-module $A/\mathfrak{p}^k$ and by $A(\mathfrak{p}^\infty)$ the $\mathfrak{p}$-primary component of the torsion A-module K/A. A $\mathfrak{p}$-primary A-module T satisfying the descending chain condition on submodules is the direct sum of a finite number of submodules of the form $A(\mathfrak{p}^k)$, $k \in \mathbf{N} \cup \{\infty\}$. The number of direct summands is independent of the decomposition of T. It is called the rank of T and is denoted by $r(T)$. If T is any torsion A-module whose $\mathfrak{p}$-primary components $T_{(\mathfrak{p})}$ satisfy the d.c.c., we set $r(T) = \sup_{\mathfrak{p} \in \mathcal{P}} r(T_{(\mathfrak{p})})$ and still call it the rank of T. When $r(T) \leq 1$, we thus have $T_{(\mathfrak{p})} \simeq A(w_{\mathfrak{p}}(T))$, where $w_{\mathfrak{p}}(T) \in \overline{\mathbf{N}}$. We denote by $w(T)$ the superdivisor defined by $w(T)(\mathfrak{p}) = w_{\mathfrak{p}}(T)$.

Unless otherwise explicitly mentioned, we further use the terminology and notations of [1], Chap. I.

(*) Nella seduta del 15 novembre 1975.

2. INVARIANTS

Let E be a three-dimensional vector space over K. Let M be a rank three A-submodule of E and let x_1, x_2, x_3 be independent elements of M.

For any x in M, let $h_{\mathfrak{p}}^{M}(x) = \sup\{k \in \mathbf{N} \cup \{0\}; \pi^{-k} x \in M_{\mathfrak{p}}\}$ and consider the superdivisor $h(M, x): \mathfrak{p} \mapsto h_{\mathfrak{p}}^{M}(x)$. In particular, set $\mu_i = h(M, x_i)$, $i = 1, 2, 3$.

Let N_i be the pure A-submodule of M generated by x_i, $i = 1, 2, 3$, and set $M/N_1 + N_2 + N_3 = M^0$. We have proved in [2] that M^0 is a torsion module and that $M^0_{(\mathfrak{p})}$ is of the form $A(\mathfrak{p}^{\mu(\mathfrak{p})}) \oplus A(\mathfrak{p}^{\mu'(\mathfrak{p})})$, with $\mu(\mathfrak{p}), \mu'(\mathfrak{p}) \in \overline{\mathbf{N}}$, and requiring $\mu'(\mathfrak{p}) \leq \mu(\mathfrak{p})$, we have thus determined two superdivisors μ and μ' which characterize the structure of M^0 and were proved to satisfy:

(C_1) $\mu' \mid \mu$;

(C_2) if there exists i such that $\mu_i(\mathfrak{p}) = \infty$, then $\mu'(\mathfrak{p}) = 0$;

(C_3) if there exist i and j, $i \neq j$, such that $\mu_i(\mathfrak{p}) = \mu_j(\mathfrak{p}) = \infty$, then $\mu(\mathfrak{p}) = \mu'(\mathfrak{p}) = 0$.

Let N_{ij} be the pure A-submodule of M generated by x_i and x_j, where $i, j = 1, 2, 3$ and $i \neq j$. Then $N_{ij}/N_i + N_j$ is a torsion module of rank at most one (see [2]). Set $h_k = w(N_{ij}/N_i + N_j)$, where $k = 1, 2, 3$ and $\{i, j, k\} = \{1, 2, 3\}$. Denote by f the canonical homomorphism of M onto M^0 and let $N^0_{ij} = f(N_{ij})$.

For $i = 1, 2, 3$, denote by M_i the A-submodule of E consisting of the elements rx_i for which there exist $s, t \in K$ such that $rx_i + sx_j + tx_k \in M$, with $\{i, j, k\} = \{1, 2, 3\}$. For any subset $\{i, j\}$ of $\{1, 2, 3\}$, denote by M_{ij} the A-submodule of E consisting of the elements $rx_i + sx_j$ for which there exists $t \in K$ such that $rx_i + sx_j + tx_k$ belongs to M, with $k \neq i, j$.

Finally, if H is a submodule of M^0 of rank at most 1, set $m_i(H, \mathfrak{p}) = w_{\mathfrak{p}}(H \cap N^0_{jk})$, with $i = 1, 2, 3$ and $\{i, j, k\} = \{1, 2, 3\}$, and let $m_i(H): \mathfrak{p} \mapsto m_i(H, \mathfrak{p})$.

2.1. LEMMA. *Let* $z_i(\mathfrak{p}) = 0$, *if* $\mu_i(\mathfrak{p}) = \infty$ *and* $z_i(\mathfrak{p}) = s_i(\mathfrak{p}) x_i$, *if* $\mu_i(\mathfrak{p}) < \infty$, *where* $s_i(\mathfrak{p}) \in K$ *satisfies* $v_{\mathfrak{p}}(s_i(\mathfrak{p})) = -\mu_i(\mathfrak{p})$ *and* $v_{\mathfrak{q}}(s_i(\mathfrak{p})) \geq 0$, *for* $\mathfrak{q} \neq \mathfrak{p}$, *and where* $i = 1, 2, 3$.

(a) *If* H *is a rank* 1 *submodule of* M^0 *and if* $0 < m \leq w_{\mathfrak{p}}(H)$, *there exist* $a_1, a_2, a_3 \in A$ *such that* $v_{\mathfrak{p}}(a_i) = \inf(m_i(H, \mathfrak{p}), m)$ *and* $h_{\mathfrak{p}}^{M}(a_1 z_1(\mathfrak{p}) + a_2 z_2(\mathfrak{p}) + a_3 z_3(\mathfrak{p})) \geq m$.

(b) *If* $b_1, b_2, b_3 \in A$ *are such that* $v_{\mathfrak{p}}(b_i) = v_{\mathfrak{p}}(a_i)$, *then* $h_{\mathfrak{p}}^{M}(b_1 z_1(\mathfrak{p}) + b_2 z_2(\mathfrak{p}) + b_3 z_3(\mathfrak{p})) \geq m$ *if and only if* $v_{\mathfrak{p}}(a_i b_j - a_j b_i) \geq m$, *for all* $\{i, j\} \subset \{1, 2, 3\}$.

To abbreviate our notations, we set $z_i(\mathfrak{p}) = z_i$ and $m_i(H, \mathfrak{p}) = m_i$. Since $r(H) = 1$, there exists at most one index i such that $m_i \neq 0$. Suppose

$m_1 \geq m_2 = m_3 = 0$. Then, by virtue of [2], Prop. 5, Cor. 1, we have $\mu_2(\mathfrak{p}), \mu_3(\mathfrak{p}) < \infty$. If $\mu_1(\mathfrak{p}) = \infty$, the lemma results immediately from [3], Lemma 1, applied to the submodule N_{23}. Suppose thus $\mu_1(\mathfrak{p}) < \infty$. If $0 < m \leq w_{\mathfrak{p}}(H)$, let $\bar{z}$ be an element of H whose order ideal is $\mathfrak{p}^m$ and let $z = r_1 z_1 + r_2 z_2 + r_3 z_3 \in f^{-1}(\bar{z})$. Then $v_{\mathfrak{p}}(r_i) \geq -m$, where equality holds for at least two values of i since otherwise the order ideal of $\bar{z}$ would contain $\mathfrak{p}^{m-1}$. We now show that z can be chosen such that $v_{\mathfrak{p}}(r_i) = \inf(m_i - m, 0)$.

If $m_1 \geq m$, we have $\bar{z} \in H \cap N_{23}^0$ and $f^{-1}(\bar{z}) \subset N_1 + N_{23}$. Then, for any $z \in f^{-1}(\bar{z})$, we have $v_{\mathfrak{p}}(r_1) \geq 0$ and z can clearly be chosen such that $v_{\mathfrak{p}}(r_1) = 0$.

Now let $m_1 < m$. Let $z = r_1 z_1 + r_2 z_2 + r_3 r_3 \in f^{-1}(\bar{z})$ and write $v_{\mathfrak{p}}(r_1) = h - m$. Therefore $\mathfrak{p}^{m-h} \bar{z} \subset H \cap N_{23}^0$ and hence $h \leq m_1$. If we had $h < m_1$, there would exist $\bar{y} \in H \cap N_{23}^0$ of order $\mathfrak{p}^{h+1}$ and $a \in A$ such that $\bar{y} = a\bar{z}$ and $v_{\mathfrak{p}}(a) = m - h - 1$. On the other hand, we would have $az \in N_1 + N_{23}$, which implies $v_{\mathfrak{p}}(ar_1) \geq 0$ and $v_{\mathfrak{p}}(a) \geq m - h$. Thus $h = m_1$.

Now, if $c \in A$ is such that $v_{\mathfrak{p}}(c) = m$ and $v_{\mathfrak{q}}(c) \geq \sup(0, -v_{\mathfrak{q}}(r_1), -v_{\mathfrak{q}}(r_2), -v_{\mathfrak{q}}(r_3))$, for $\mathfrak{q} \neq \mathfrak{p}$, then the $a_i = cr_i$, $i = 1, 2, 3$, satisfy part (a) of the lemma.

To prove (*b*), suppose first that $v_{\mathfrak{p}}(a_i b_j - a_j b_i) \geq m$, for all $\{i, j\} \subset \subset \{1, 2, 3\}$. Then, from [3], Lemmas 1 and 2, applied to the module M_{23}, we know that if $r \in K$ is such that $v_{\mathfrak{p}}(r) = -m$ and $v_{\mathfrak{q}}(r) \geq 0$, for $\mathfrak{q} \neq \mathfrak{p}$, there exist $s \in K$ and $d_2, d_3 \in A$ such that $r(b_2 z_2 + b_3 z_3) = s(a_2 z_2 + a_3 z_3) + d_2 z_2 + d_3 z_3$, with $v_{\mathfrak{p}}(s) \geq -m$ and $v_{\mathfrak{q}}(s) \geq 0$, if $\mathfrak{q} \neq \mathfrak{p}$. Thus $rb_2 = sa_2 + d_2$ and $rb_3 = sa_3 + d_3$. Set $d_1 = rb_1 - sa_1$. Then $s(a_3 b_1 - a_1 b_3) = d_3 b_3 - d_1 b_1$ and thus $d_1 \in A$. Therefore, $r(b_1 z_1 + b_2 z_2 + b_3 z_3) = s(a_1 z_1 + a_2 z_2 + a_3 z_3) + d_1 z_1 + d_2 z_2 + d_3 z_3$ belongs to M.

Conversely, let $h_{\mathfrak{p}}^{M}(b_1 z_1 + b_2 z_2 + b_3 z_3) \geq m$. There exist $d_1, d_2, d_3 \in A$ and $r, s \in K$ such that $v_{\mathfrak{p}}(r), v_{\mathfrak{p}}(s) \geq -m$, $v_{\mathfrak{q}}(r), v_{\mathfrak{q}}(s) \geq 0$, if $\mathfrak{q} \neq \mathfrak{p}$, and $r(b_1 z_1 + b_2 z_2 + b_3 z_3) = s(a_1 z_1 + a_2 z_2 + a_3 z_3) + d_1 z_1 + d_2 z_2 + d_3 z_3$. This, together with $m_2 = m_3 = 0$, implies $v_{\mathfrak{p}}(a_2 b_3 - a_3 b_2) \geq m$ (see [3]). We thus have $v_{\mathfrak{p}}(a_3 b_1 - a_1 b_3) = h_{\mathfrak{p}}^{M}((a_3 b_1 - a_1 b_3) z_1) = h_{\mathfrak{p}}^{M}((b_1 z_1 + b_2 z_2 + b_3 z_3) a_3 - (a_1 z_1 + a_2 z_2 + a_3 z_3) b_3 + (a_2 b_3 - a_3 b_2) z_2) \geq m$. Similarly, we would obtain $v_{\mathfrak{p}}(a_2 b_1 - a_1 b_2) \geq m$.

2.2. COROLLARY. *Let* $M^0 = H \oplus H'$, *with* $w(H) = \mu$ *and* $w(H') = \mu'$. *For all* $0 < m \leq \mu(\mathfrak{p})$ *and* $0 < m' \leq \mu'(\mathfrak{p})$, *choose* $a_i = a_i(\mathfrak{p}, m)$ *and* $a'_i = a'_i(\mathfrak{p}, m)$ *in* A, *corresponding respectively to* H *and* H', *and satisfying Lemma 2.1. Then* $v_{\mathfrak{p}}(a_i a'_j - a_j a'_i) = 0$, *for all* $\{i, j\} \subset \{1, 2, 3\}$.

This results from the fact that $H \cap H' = 0$.

If $0 < m \leq \mu(\mathfrak{p})$ (resp. $0 < m \leq \mu'(\mathfrak{p})$), set $y(\mathfrak{p}, m) = t(\mathfrak{p}, m)(a_1(\mathfrak{p}, m) z_1(\mathfrak{p}) + a_2(\mathfrak{p}, m) z_2(\mathfrak{p}) + a_3(\mathfrak{p}, m) z_3(\mathfrak{p}))$ (resp. $y'(\mathfrak{p}, m) = t(\mathfrak{p}, m)(a'_1(\mathfrak{p}, m) z_1(\mathfrak{p}) + a'_2(\mathfrak{p}, m) z_2(\mathfrak{p}) + a'_3(\mathfrak{p}, m) z_3(\mathfrak{p}))$, with $v_{\mathfrak{p}}(t(\mathfrak{p}, m)) = -m$ and $v_{\mathfrak{q}}(t(\mathfrak{p}, m)) \geq 0$, for $\mathfrak{q} \neq \mathfrak{p}$. Thus $y(\mathfrak{p}, m) \in f^{-1}(H)$ (resp. $y'(\mathfrak{p}, m) \in f^{-1}(H')$).

If $\mu_i(\mathfrak{p}) = \infty$, then for each $n \in \mathbf{N}$, let $t_i(\mathfrak{p}, n) \in K$ be such that $v_\mathfrak{p}(t_i(\mathfrak{p}, n)) = -n$ and $v_\mathfrak{q}(t_i(\mathfrak{p}, n)) \geq 0$ for $\mathfrak{q} \neq \mathfrak{p}$. Set $y_i(\mathfrak{p}, n) = t_i(\mathfrak{p}, n) x_i$, $i = 1, 2, 3$.

2.3. LEMMA. *Let μ_1, μ_2, μ_3, μ and μ' be the superdivisors associated with (M, x_1, x_2, x_3). Let G be the set consisting of the x_i's, the $z_i(\mathfrak{p})$'s for all $\mathfrak{p}$ such that $\mu_i(\mathfrak{p}) < \infty$, the $y_i(\mathfrak{p}, n)$'s for all $\mathfrak{p}$ such that $\mu_i(\mathfrak{p}) = \infty$ and all $n \in \mathbf{N}$, the $y(\mathfrak{p}, m)$'s for all $\mathfrak{p}$ such that $\mu(\mathfrak{p}) \neq 0$ and all $m \in \mathbf{N}$ such that $m \leq \mu(\mathfrak{p})$ and, finally, the $y'(\mathfrak{p}, m)$'s for all $\mathfrak{p}$ such that $\mu'(\mathfrak{p}) \neq 0$ and all $m \in \mathbf{N}$ such that $m \leq \mu'(\mathfrak{p})$. Then G is a generating system of M.*

Indeed, let N be the submodule of M generated by G. Then $f(N) = f(M) = M^0$, since the images of the $y(\mathfrak{p}, m)$'s and the $y'(\mathfrak{p}, m)$'s in M^0 generate M^0. Therefore, $M \subset N + \ker f$. But, the x_i's, $z_i(\mathfrak{p})$'s and $y_i(\mathfrak{p}, n)$'s generate $\ker f$ and thus we have $\ker f \subset N$ and $M = N$.

2.4. LEMMA. *Let H be a rank 1 submodule of M^0 such that $w(H) = \mu$. Then $h_i = [\mu, \mu' m_i(H)]$.*

We must prove that $w_\mathfrak{p}(N^0_{jk}) = \inf(\mu(\mathfrak{p}), \mu'(\mathfrak{p}) + m_i(H, \mathfrak{p}))$, where $m_i(H, \mathfrak{p}) = w_\mathfrak{p}(H \cap N^0_{jk})$. This is obvious when $\mu'(\mathfrak{p}) = 0$ or when $\mu'(\mathfrak{p}) = \mu(\mathfrak{p})$. Suppose thus $0 < \mu'(\mathfrak{p}) < \mu(\mathfrak{p}) \leq \infty$, and suppose as before that $m_1 \geq m_2 = m_3 = 0$, where m_i stands for $m_i(H, \mathfrak{p})$. We have immediately $h_2(\mathfrak{p}) = h_3(\mathfrak{p}) = \mu'(\mathfrak{p})$, and it remains to show that $h_1(\mathfrak{p}) = \inf(\mu(\mathfrak{p}), \mu'(\mathfrak{p}) + m_1)$. This equality is obvious if $m_1 = \mu(\mathfrak{p})$. Let then $m_1 < \mu(\mathfrak{p})$ and let $m \in \mathbf{N}$ such that $\sup(m_1, \mu'(\mathfrak{p})) < m \leq \mu(\mathfrak{p})$. The submodule H is a direct summand of M^0; consider H' such that $M^0 = H \oplus H'$ and let $y(\mathfrak{p}, m)$ and $y'(\mathfrak{p}, \mu'(\mathfrak{p}))$ be defined as in Lemma 2.3. To simplify the notations, set $y(\mathfrak{p}, m) = y(m) = t(m)(a_1(m) z_1 + a_2(m) z_2 + a_3(m) z_3)$ and $y'(\mathfrak{p}, \mu'(\mathfrak{p})) = y' = t'(a_1' z_1 + a_2' z_2 + a_3' z_3)$. By Corollary 2.2., $m_1 = v_\mathfrak{p}(a_1(m)) > 0$ implies $v_\mathfrak{p}(a_1') = 0$, i.e. at least one of the ideals $a_1(m) A$ and $a_1' A$ is comaximal with $\mathfrak{p}^{\mu'(\mathfrak{p})}$. Then, there exist $b, c, d \in A$ such that $b a_1(m) + c a_1' + d = 0$, with $v_\mathfrak{p}(b) = v_\mathfrak{p}(a_1')$, $v_\mathfrak{p}(c) = m_1$ and $v_\mathfrak{p}(d) = \mu'(\mathfrak{p}) + m_1$. Therefore, $b(a_1(m) z_1 + a_2(m) z_2 + a_3(m) z_3) + c(a_1' z_1 + a_2' z_2 + a_3' z_3) + dz_1 = (b a_2(m) + c a_2') z_2 + (b a_3(m) + c a_3') z_3 = u(m) \in N_{23}$, with $h^M_\mathfrak{p}(u(m)) \geq \inf(m, \mu'(\mathfrak{p}) + m_1)$ and thus $h_1(\mathfrak{p}) \geq \inf(\mu(\mathfrak{p}), \mu'(\mathfrak{p}), \mu'(\mathfrak{p}) + m_1)$. The lemma is proved if $\mu'(\mathfrak{p}) + m_1 \geq \mu(\mathfrak{p})$. It remains to consider the case where $\mu'(\mathfrak{p}) + m_1 < \mu(\mathfrak{p})$ and to show that $h_1(\mathfrak{p}) \leq \mu'(\mathfrak{p}) + m_1$.

Suppose, on the contrary, that there exists $b z_2 + c z_3 \in N_{23}$ such that $v_\mathfrak{p}(b) = v_\mathfrak{p}(c) = 0$ and $h^M_\mathfrak{p}(b z_2 + c z_3) = \mu'(\mathfrak{p}) + m_1 + 1$. Then, by Lemma 2.3, we can find $k \in K$, such that $v_\mathfrak{p}(k) = -\mu'(\mathfrak{p}) - m_1 - 1$ and $v_\mathfrak{q}(k) \geq 0$ for $\mathfrak{q} \neq \mathfrak{p}$, and $d, d', n_1, n_2, n_3 \in A$ satisfying $k(b z_2 + c z_3) = \mathrm{d} y(m) + d' y' + n_1 z_1 + n_2 z_2 + n_3 z_3$, where $m = \mu'(\mathfrak{p}) + m_1 + 1$. This means $\mathrm{d} t(m) a_1(m) + d' t' a_1' + n_1 = 0$, $\mathrm{d} t(m) a_2(m) + d' t' a_2' + n_2 = kb$ and $\mathrm{d} t(m) a_3(m) + d' t' a_3' + n_3 = kc$. But, the first equality inplies $v_\mathfrak{p}(d) > 0$, while each of the last two implies $v_\mathfrak{p}(d) = 0$!

2.5. COROLLARY. *If* $h_i(\mathfrak{p}) < \mu(\mathfrak{p})$ *for all* $i = 1, 2, 3$, *then for all* $\{j, k\} \subset \{1, 2, 3\}$, $w_\mathfrak{p}(\mathrm{H} \cap \mathrm{N}^0_{jk})$ *is independent of the choice of* H, *provided* $w(\mathrm{H}) = \mu$.

2.6. DEFINITION. We shall use the term *adele*, in a restricted sense, to designate the elements of the product ring $\mathscr{A} = \prod_{\mathfrak{p} \in \mathscr{P}} \bar{\mathrm{A}}_\mathfrak{p}$, where $\bar{\mathrm{A}}_\mathfrak{p}$ is the completion of A with respect to the discrete valuation $v_\mathfrak{p}$. We shall identify the element a of A with the adele $(a(\mathfrak{p}))$ defined by letting $a(\mathfrak{p}) = a$ for all $\mathfrak{p} \in \mathscr{P}$. For any $\mathfrak{p} \in \mathscr{P}$ and $\eta \in \mathscr{A}$, we set $v_\mathfrak{p}(\eta) = v_\mathfrak{p}(\eta(\mathfrak{p}))$ and $v(\eta) : \mathfrak{p} \mapsto v_\mathfrak{p}(\eta)$.

Let μ be a superdivisor and let $(\eta_1, \eta_2, \eta_3), (\eta_1', \eta_2', \eta_3') \in \mathscr{A}^3$ such that for all $\mathfrak{p} \in \mathscr{P}$,

$$\inf_{\substack{i,j \in \{1,2,3\} \\ i \neq j}} v_\mathfrak{p}(\eta_i \eta_j) = \inf_{\substack{i,j \in \{1,2,3\} \\ i \neq j}} v_\mathfrak{p}(\eta_i' \eta_j') = 0 .$$

We shall say that (η_1, η_2, η_3) *and* $(\eta_1', \eta_2', \eta_3')$ *are* μ*-equivalent* (in symbol $(\eta_1, \eta_2, \eta_3) \underset{\mu}{\sim} (\eta_1', \eta_2', \eta_3')$) if

(1) $v(\eta_i) = v(\eta_i')$, $i = 1, 2, 3$ and

(2) $\mu \mid v(\eta_i \eta_j' - \eta_j \eta_i')$, $i, j = 1, 2, 3, i \neq j$.

2.7. LEMMA. (a) *Let* $z_i(\mathfrak{p})$ *be defined as in Lemma 2.1. and let* H *be a submodule of* M^0 *of rank at most 1. Then, there exists* $(\eta_1, n_2, \eta_3) \in \mathscr{A}^3$ *such that*

(i) $v(\eta_i) = m_i(\mathrm{H})$, *for* $i = 1, 2, 3$;

(ii) *if* $w_\mathfrak{p}(\mathrm{H}) \neq 0$, *if* $m \in \mathbf{N}$ *is such that* $m \leq w_\mathfrak{p}(\mathrm{H})$ *and if* $a_1, a_2, a_3 \in \mathrm{A}$ *satisfy* $v_\mathfrak{p}(a_i) = \inf(m_i(\mathrm{H}, \mathfrak{p}), m)$, *then* $h_\mathfrak{p}^\mathrm{M}(a_1 z_1(\mathfrak{p}) + a_2 z_2(\mathfrak{p}) + a_3 z_3(\mathfrak{p})) \geq m$ *if and only if* $v_\mathfrak{p}(a_i \eta_j - a_j \eta_i) \geq m$, *for all* $i, j = 1, 2, 3, i \neq j$.

(b) *A triple* $(n_1^0, \eta_2^0, \eta_3^0)$ *satisfies* (i) *and* (ii) *if and only if* $(\eta_1^0, \eta_2^0, \eta_3^0) \underset{w(\mathrm{H})}{\sim} (\eta_1, \eta_2, \eta_3)$.

Clearly, we can suppose immediately that $r(\mathrm{H}) = 1$. Consider a fixed $\mathfrak{p}$ such that $w_\mathfrak{p}(\mathrm{H}) \neq 0$. It suffices to show the existence of $\eta_1(\mathfrak{p}), \eta_2(\mathfrak{p}), \eta_3(\mathfrak{p}) \in \bar{\mathrm{A}}_\mathfrak{p}$ satisfying conditions (i) and (ii).

As before, suppose $m_2 = m_3 = 0$. The existence of η_1, η_2, η_3 is obvious when $w_\mathfrak{p}(\mathrm{H}) < \infty$ and results from [3], Lemma 3, when $w_\mathfrak{p}(\mathrm{H}) = m_1 = \infty$. Now suppose $w_\mathfrak{p}(\mathrm{H}) = \infty$ and $m_1 < \infty$. Consider the sequences $(a_1(m))$, $(a_2(m))$ and $(a_3(m))$ of elements of A, with $m \geq m_1$, as defined in Lemma 2.1. We thus have $v_\mathfrak{p}(a_1(m)) = m_1$ and $v_\mathfrak{p}(a_2(m)) = v_\mathfrak{p}(a_3(m)) = 0$. The ideals $a_3(m)\mathrm{A}$ and $\mathfrak{p}^m$ being comaximal, there exist $c_m \in \mathrm{A} - \mathfrak{p}$ and $d_m \in \mathfrak{p}^m$ satisfying $c_m a_3(m) + d_m = 1$. Let $b_i(m) = c_m a_i(m)$. Then $h_\mathfrak{p}^\mathrm{M}(c_m(a_1(m) z_1 + a_2(m) z_2 + a_3(m) z_3) + d_m z_3) = h_\mathfrak{p}^\mathrm{M}(b_1(m) z_1 + b_2(m) z_2 + z_3) \geq m$. Similarly, we have $h_\mathfrak{p}^\mathrm{M}(b_1(m+1) z_1 + b_2(m+1) z_2 + z_3) \geq m + 1$, and applying again Lemma 2.1., we obtain $v_\mathfrak{p}(b_1(m+1) - b_1(m)) \geq m$ and $v_\mathfrak{p}(b_2(m+1) - b_2(m)) \geq m$. The sequences $(b_1(m))$ and $(b_2(m))$ are thus converging in $\bar{A}_{\mathfrak{p}}$, say to η_1 and η_2. Now, take $\eta_3 = 1$. It is readily checked that η_1, η_2, η_3 satisfy (i) and (ii).

Part (b) is the result of an easy calculation which we omit.

2.7. COROLLARY. *Let* $M^0 = H \oplus H'$, *with* $w(H) = \mu$ *and* $w(H') = \mu'$. *Let* $(\eta_1, \eta_2, \eta_3), (\eta'_1, \eta'_2, \eta'_3) \in \mathscr{A}^3$ *correspond respectively to* H *and* H'. *Then, if* $\mu'(\mathfrak{p}) \neq 0$, *we have* $v_{\mathfrak{p}}(\eta_i \eta'_i - \eta_j \eta'_j) = 0$ *for all* $i, j = 1, 2, 3, i \neq j$.

This results immediately from the previous lemma and Cor. 2.2.

2.9. DEFINITION. Let Q and Q′ be respectively a μ-class and a μ'-class of elements of $\mathscr{A}^3$. We shall say that Q *and* Q′ *are compatible* if, whenever we have $\mu'(\mathfrak{p}) \neq 0$, then $v_{\mathfrak{p}}(\eta_i \eta'_i - \eta_j \eta'_j) = 0$ for all $(\eta_1, \eta_2, \eta_3) \in Q$, $(\eta'_1, \eta'_2, \eta'_3) \in Q'$ and $\{i, j\} \subset \{1, 2, 3\}$.

For each decomposition $H \oplus H'$ of M^0, Corollary 2.8. ensures the existence of a pair (Q, Q') consisting of a μ-class and a μ'-class which are compatible. We shall now investigate the relations between two pairs (Q, Q') and $(\bar{Q}, \bar{Q}')$ associated with distinct decompositions $H \oplus H'$ and $\bar{H} \oplus \bar{H}'$ of M^0.

2.10. LEMMA. *Let* $H \oplus H'$ *and* $\bar{H} \oplus \bar{H}'$ *be two decompositions of* M^0, *with* $w(H) = w(\bar{H}) = \mu$ *and* $w(H') = w(\bar{H}') = \mu'$. *Let* Q *and* $\bar{Q}$ *(resp.* Q′ *and* $\bar{Q}'$*) be the corresponding* μ*-classes (resp.* μ'*-classes). Then, for every* $(\eta_1, \eta_2, \eta_3) \in Q$ *and* $(\eta'_1, \eta'_2, \eta'_3) \in Q'$, *there exists a matrix* $\begin{pmatrix} \alpha & \alpha' \\ \beta & \beta' \end{pmatrix}$ *with coefficients in* $\mathscr{A}$ *and such that*

(1) $(\alpha\eta_1 + \alpha'\eta'_1, \alpha\eta_2 + \alpha'\eta'_2, \alpha\eta_3 + \alpha'\eta'_3) \in \bar{Q}$.

(2) $(\beta\eta_1 + \beta'\eta'_1, \beta\eta_2 + \beta'\eta'_2, \beta\eta_3 + \beta'\eta'_3) \in \bar{Q}'$,

(3) $v(\alpha\beta' - \alpha'\beta) = 1$,

(4) $\mu \mid \mu'(v(\alpha'))$.

Let $\mathfrak{p}$ be a fixed non-zero prime ideal.

Case 1: $\mu'(\mathfrak{p}) \leq \mu(\mathfrak{p}) < \infty$. Let $\eta_i(\mathfrak{p}) = a_i + \xi_i$, with $v_{\mathfrak{p}}(\xi_i) \geq \mu(\mathfrak{p})$, and let $\eta'_i(\mathfrak{p}) = a'_i + \xi'_i$, with $v_{\mathfrak{p}}(\xi'_i) \geq \mu'(\mathfrak{p})$, where $a_i, a'_i \in A$. Set $y(\mathfrak{p}, \mu(\mathfrak{p})) = y = t(a_1 z_1 + a_2 z_2 + a_3 z_3)$ and $y'(\mathfrak{p}, \mu'(\mathfrak{p})) = y' = t'(a'_1 z_1 + a'_2 z_2 + a'_3 z_3)$. Choose $(\bar{a}_1, \bar{a}_2, \bar{a}_3) \in \bar{Q}$ and $(\bar{a}'_1, \bar{a}'_2, \bar{a}'_3) \in \bar{Q}'$ with $\bar{a}_i, \bar{a}'_i \in A$. Set $\bar{y}(\mathfrak{p}, \mu(\mathfrak{p})) = \bar{y} = t(\bar{a}_1 z_1 + \bar{a}_2 z_2 + \bar{a}_3 z_3)$ and $\bar{y}'(\mathfrak{p}, \mu'(\mathfrak{p})) = \bar{y}' = t'(\bar{a}'_1 z_1 + \bar{a}'_2 z_2 + \bar{a}'_3 z_3)$.

There exist $c, c', r_1, r_2, r_3 \in A$ such that $\bar{y} = cy + c'y' + r_1 z_1 + r_2 z_2 + r_3 z_3$ and similarly, there exist $d, d', s_1, s_2, s_3 \in A$ such that $\bar{y}' = dt' t^{-1} y + d'y' + s_1 z_1 + s_2 z_2 + s_3 z_3$. Thus $\bar{a}_i = ca_i + c' t' t^{-1} a'_i + t^{-1} r_i$ and $\bar{a}'_i = da_i + d' a'_i + t'^{-1} s_i$. Let $\alpha(\mathfrak{p}) = c$, $\alpha'(\mathfrak{p}) = c' t' t^{-1}$, $\beta(\mathfrak{p}) = d$ and $\beta'(\mathfrak{p}) = d'$. Then $v_{\mathfrak{p}}(\alpha'(\mathfrak{p})) \geq \mu(\mathfrak{p}) - \mu'(\mathfrak{p})$ and $v_{\mathfrak{p}}((\alpha(\mathfrak{p}) a_i + \alpha'(\mathfrak{p}) a'_i) \bar{a}_j - (\alpha(\mathfrak{p}) a_j + \alpha'(\mathfrak{p}) a'_j) \bar{a}_i) \geq \mu(\mathfrak{p})$ and $v_{\mathfrak{p}}((\beta(\mathfrak{p}) a_i + \beta'(\mathfrak{p}) a'_i) \bar{a}_j - (\beta(\mathfrak{p}) a_j + \beta'(\mathfrak{p}) \alpha'_j) \bar{a}_i) \geq \mu'(\mathfrak{p})$. These relations imply $v_{\mathfrak{p}}((\alpha(\mathfrak{p}) \eta_i(\mathfrak{p}) + \alpha'(\mathfrak{p}) \eta'_i(\mathfrak{p})) \bar{a}_j - (\alpha(\mathfrak{p}) \eta_j(\mathfrak{p}) + \alpha'(\mathfrak{p}) \eta'_j(\mathfrak{p})) \bar{a}_i) \geq \mu(\mathfrak{p})$ and $v_{\mathfrak{p}}((\beta(\mathfrak{p}) \eta_i(\mathfrak{p}) + \beta'(\mathfrak{p}) \eta'_i(\mathfrak{p})) \bar{a}_j - (\beta(\mathfrak{p}) \eta_j(\mathfrak{p}) + \beta'(\mathfrak{p}) \eta'_j(\mathfrak{p}) \bar{a}_i) \geq \mu'(\mathfrak{p})$, which proves (1) and (2) at $\mathfrak{p}$. In addition, the relations giving $\bar{y}$ and $\bar{y}'$ in terms of y and y' must be invertible and therefore $v_{\mathfrak{p}}(cd' - c'\,dt'\,t^{-1}) = v_{\mathfrak{p}}(\alpha(\mathfrak{p})\,\beta'(\mathfrak{p}) - \alpha'(\mathfrak{p})\,(\beta)(\mathfrak{p})) = 0$.

Case 2: $\mu'(\mathfrak{p}) < \infty$ *and* $\mu(\mathfrak{p}) = \infty$. Then $H_{(\mathfrak{p})}$ is the largest divisible submodule of $M^0_{(\mathfrak{p})}$ and therefore $\bar{H}_{(\mathfrak{p})} = H_{(\mathfrak{p})}$. We can thus choose $\alpha(\mathfrak{p}) = 1$ and $\alpha'(\mathfrak{p}) = 0$. The existence of $\beta(\mathfrak{p})$ and $\beta'(\mathfrak{p})$ is proved as before.

Case 3: $\mu'(\mathfrak{p}) = \mu(\mathfrak{p}) = \infty$. Let $(a_i(m))$ and $(a_i'(m))$ be sequences of elements of A defined as in Lemma 2.1. and converging respectively to $\eta_i(\mathfrak{p})$ and $\eta_i'(\mathfrak{p})$. On the other hand, let $(\bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3) \in \bar{Q}$ and $(\bar{\eta}_1', \bar{\eta}_2', \bar{\eta}_3') \in \bar{Q}'$. Let $(\bar{a}_i(m))$ and $(\bar{a}_i'(m))$ be sequences of elements of A converging respectively to $\bar{\eta}_i(\mathfrak{p})$ and $\bar{\eta}_i'(\mathfrak{p})$. Proceeding as in Case 1, we can find for every $m \in \mathbf{N}$ elements c_m, c_m', d_m, d_m' of A such that $\inf(v_{\mathfrak{p}}(c_m), v_{\mathfrak{p}}(c_m')) = \inf(v_{\mathfrak{p}}(d_m), v_{\mathfrak{p}}(d_m')) = 0$, $v_{\mathfrak{p}}((c_m a_i(m) + c_m' a_i'(m))\,\bar{a}_j(m) - (c_m a_j(m) + c_m' a_j'(m))\,\bar{a}_i(m)) \geq m$ and $v_{\mathfrak{p}}((d_m a_i(m) + d_m' a_i'(m))\,\bar{a}_j(m) - (d_m a_j(m) + d_m' a_j'(m))\,\bar{a}_i(m)) \geq m$. Moreover, taking into account Cor. 2.2., it is easy to prove that $\inf(v_{\mathfrak{p}}(c_m), v_{\mathfrak{p}}(d_m)) = \inf(v_{\mathfrak{p}}(c_m'), v_{\mathfrak{p}}(d_m')) = 0$. Then, assuming for example that $v_{\mathfrak{p}}(c_m) = v_{\mathfrak{p}}(d_m') = 0$, we still have $v_{\mathfrak{p}}(c_{m+k}) = v_{\mathfrak{p}}(d_{m+k}') = 0$ for every $k \in \mathbf{N}$, and it is readily checked that c_m and d_m' can be taken equal to 1 for all $m \in \mathbf{N}$. We now have $v_{\mathfrak{p}}((a_i(m) + c_m' a_i'(m))\,\bar{a}_j(m) - (a_j(m) + c_m' a_j'(m))\,\bar{a}_i(m)) \geq m$ and clearly also $v_{\mathfrak{p}}((a_i(m) + c_{m+1}' a_i'(m))\,\bar{a}_j(m) - (a_j(m) + c_{m+1}' a_j'(m))\,\bar{a}_i(m)) \geq m$. These relations imply $v_{\mathfrak{p}}(c_{m+1}' - c_m') \geq m$, i.e. the sequence (c_m') has a limit $\alpha'(\mathfrak{p})$ in $\bar{A}_{\mathfrak{p}}$. Similarly, the sequence (d_m) converges to an element $\beta(\mathfrak{p})$ of $\bar{A}_{\mathfrak{p}}$ and, choosing $\alpha(\mathfrak{p}) = \beta'(\mathfrak{p}) = 1$, we obtain

$$(\alpha(\mathfrak{p})\,\eta_i(\mathfrak{p}) + \alpha'(\mathfrak{p})\,\eta_i'(\mathfrak{p}))\,\bar{\eta}_j(\mathfrak{p}) = (\alpha(\mathfrak{p})\,\eta_j(\mathfrak{p}) + \alpha'(\mathfrak{p})\,\eta_j'(\mathfrak{p}))\,\bar{\eta}_i(\mathfrak{p}),$$

$$(\beta(\mathfrak{p})\,\eta_i(\mathfrak{p}) + \beta'(\mathfrak{p})\,\eta_i'(\mathfrak{p}))\,\bar{\eta}_j(\mathfrak{p}) = (\beta(\mathfrak{p})\,\eta_j(\mathfrak{p}) + \beta'(\mathfrak{p})\,\eta_j'(\mathfrak{p}))\,\bar{\eta}_i(\mathfrak{p}).$$

2.11. DEFINITION. We shall say that the pairs (Q, Q') and $(\bar{Q}, \bar{Q}')$ are equivalent if, for every $(\eta_1, \eta_2, \eta_3) \in Q$ and $(\eta_1', \eta_2', \eta_3') \in Q'$, there exists a matrix $\begin{pmatrix}\alpha & \alpha'\\ \beta & \beta'\end{pmatrix}$ with coefficients in $\mathscr{A}$ and satisfying conditions (1), (2), (3) and (4) of Lemma 2.10. It is trivial to check that this defines an equivalence relation.

With (M, x_1, x_2, x_3) are thus associated the superdivisors μ_1, μ_2, μ_3, μ and μ', and a class χ of pairs (Q, Q'). We shall write $inv(M, x_1, x_2, x_3) = (\mu_1, \mu_2, \mu_3, \mu, \mu', \chi)$.

In view of Lemmas 2.3. and 2.7., the following theorem requires no further proof:

2.12. THEOREM. *Let* x_1, x_2, x_3 *be independent elements of* E. *Let* μ_1, μ_2, μ_3, μ *and* μ' *be superdivisors satisfying conditions* (C_1), (C_2) *and* (C_3) *and let* χ *be a class of pairs* (Q, Q'). *There exists one and only one rank three* A-*submodule* M *of* E, *containing* x_1, x_2, x_3 *and such that* $inv(M, x_1, x_2, x_3) = (\mu_1, \mu_2, \mu_3, \mu, \mu', \chi)$.

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