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Lattice Measures, Realcompactness and Pseudocompactness

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RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

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Presiede il Presidente della Classe BENIAMINO SEGRE

SĚZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *Lattice Measures, Realcompactness and Pseudocompactness.* Nota I di MARTIN KERNER, presentata (*) dal Corrisp. G. ZAPPA.

RIASSUNTO. — Dati uno spazio X e un reticolo $\mathscr{L}$ di sottoinsiemi, c'è una corrispondenza 1 : 1 tra le misure 0 — 1 $\mathscr{L}$–regolari e gli $\mathscr{L}$–ultrafiltri. Esaminiamo quali cambiamenti subisce la teoria quando le misure non vengono assunte regolari. Topologizziamo lo spazio delle misure $\mathscr{L}$–regolari e mostriamo che questo spazio topologico è T — 1 e compatto. Infine colleghiamo questo spazio a certi spazi topologici studiati da Alexandrof e Varadarajan.

1. INTRODUCTION

In this paper we make use of the relationship between measures and ultrafilters to use analytic machinery in a topological setting. This and similar relationships, discussed by Bachman and Cohen [3], Frolik [6], Topsøe [10], Alo and Shapiro [2] and others, allow us to study properties of the associated measure rather than those of the ultrafilter. In this setting results of Brooks [4] and Kost [9] are obtained as corollaries and we get measure theoretic characterizations of normal lattices. Also, we get a 'mirror' effect. Starting with a lattice, $\mathscr{L}$, of subsets, we generate measures on $\mathscr{A}(\mathscr{L})$, the smallest algebra containing $\mathscr{L}$. Subsets of this space of measures form a lattice, and we relate properties of this lattice to properties of the original.

We then look at lattices as a more general setting for pseudocompactness and realcompactness, concepts usually defined in terms of Baire sets or zero sets. When the underlying lattice is allowed to be more general, we

(*) Nella seduta del 15 novembre 1975.

get a result analogous to that of Glicksberg [11] for $\mathscr{L}$-pseudocompactness, and generalize a theorem of Varadarajan [11] for realcompact spaces which is itself obtained as a corollary.

2. Background and Notations

Given a space X, a collection of subsets is a lattice if it contains φ, X, and is closed under finite unions and intersections. A δ-lattice is one closed under countable unions. A T — 2 lattice is one with the following property: $x \neq y$, $x, y \in X \Rightarrow$ there exist A, B in the lattice such that $x \in A'$, $y \in B'$ and $A' \cap B' = \varphi$. Similarly, a lattice $\mathscr{L}$ is normal if, for all $L_1, L_2 \in \mathscr{L}$ there exist $A, B \in \mathscr{L}$ such that $A' \supset L_1$, $B' \supset L_2$ and $A' \cap B' = \varphi$. X is $\mathscr{L}$-compact if $X = \cup A'$, $A \in \mathscr{L}$, implies that $X = \bigcup_1^n A_j'$. Intuitively, the closed sets in a topological space form a lattice and we use this as a model to generalize notions of separation and compactness to arbitrary lattices. A lattice is atom disjunctive if, given $x \notin A \in \mathscr{L}$, there exists $B \in \mathscr{L}$ with $x \in B$, $A \cap B = \varphi$.

By a measure, μ, we shall mean a finitely additive set function defined on an algebra or σ-algebra, evaluating to zero on $\varnothing$. A countably additive measure will be called σ-smooth. We relate a measure to the lattice structure of a space with the following: A measure, μ, is $\mathscr{L}$-regular if, for all A, $\mu(A) = \sup \mu(L)$, $A \supset L$, $L \in \mathscr{L}$. In this paper we will be primarily interested in 0 — 1 measures, measures taking only two possible values.

A filter is a collection of non-empty subsets of a space X closed under finite intersection and supersets. An ultrafilter is a maximal element in the ordered set of filters.

The next results indicate the relationship between measures and filters and relate some of the analytic properties of measures to topological properties of filters. These properties have been studied by Bachman and Cohen [3] and by the author [8], and will be referred to frequently in this paper.

Given a space, X, and a lattice, $\mathscr{L}$, of subsets of X, let μ be a 0 — 1 $\mathscr{L}$-regular measure defined on $\mathscr{A}(\mathscr{L})$.

LEMMA 2.1. *Let* $\mathscr{F}_\mu = \{L \in \mathscr{L} \mid \mu(L) = 1\}$. *Than* $\mathscr{F}_\mu$ *is an* $\mathscr{L}$*-ultrafilter.*

Proof. The proof is straightforward.

Let G be an $\mathscr{L}$-ultrafilter. Define A_G as $\{E \mid E \supset L, L \in G \text{ or } E \subset L', L \in G\}$ (L' = complement of L).

LEMMA 2.2. A_G *is an algebra containing* $\mathscr{A}(\mathscr{L})$.

Proof. It is not difficult to show that A_G is an algebra. To show that it contains $\mathscr{A}(\mathscr{L})$ we use the following result about ultrafilters: Given an ultrafilter F and a set A which intersects every member of F, than $A \in F$. If $B \in \mathscr{L}$ and $B \supset \bar{L} \in G$, $B \in G$ and $B \in A_G$. If $B \in \mathscr{L}$ and $B \not\subset A'$ for any $A \in G$, than $B \cap A \neq \varphi$ for any $A \in G$ and by our result about ultrafilters $B \in G \Rightarrow B \in A_G$. Together, $A_G \supset \mathscr{L} \Rightarrow A_G \supset \mathscr{A}(\mathscr{L})$.

Define ν on A_G by $\nu(E) = 1$ if $E \supset L \in G$; $\nu(E) = 0$ if $E \subset L', L \in G$. It is easy to show that ν is well defined and a measure on A_G.

CLAIM. *ν is $\mathscr{L}$-regular.*

Proof. If $\nu(E) = 1$, then E contains $L \in G$ and $\nu(L) = 1$. If $\nu(E) = 0$ and $E \supset L \in \mathscr{L}$, $\nu(E') = 1$ $E' \supset \bar{L} \in \mathscr{L}$ and $\nu(\bar{L}) = 1$. $\nu(L)$ must then be 0. In both cases $\nu(E) = \sup \nu(L)$, $E \supset L$, $\in \mathscr{L}$.

LEMMA 2.3. $\{g \in A_G \mid \nu(g) = 1\} = G$.

Proof. If $g \in G$, $\nu(g) = 1$ by definition. Assume that $K \in A_G$, $K \notin G$.

Then $K \not\supset g \in G \Rightarrow K' \in G$. But then $\nu(K') = 1$ and $\nu(K) = 0$. We have shown that $K \notin G \Rightarrow \nu(K) = 0$ which is equivalent to $\nu(K) = 1 \Rightarrow K \in G$.

These last lemmas give the following result:

THEOREM 2.1. *There is a 1 : 1 correspondence between $\mathscr{L}$-ultrafilters and 0 — 1 $\mathscr{L}$-regular measures. Each such measur^e generates an ultrafilter and every ultrafilter generates a measure which evaluates to 1 on its own elements.*

THEOREM 2.2. *μ is σ-smooth iff $\mathscr{F}_\mu$ has the countable intersection property.*

$$(\text{i.e. } F_n \in \mathscr{F}_\mu \Rightarrow \cap F_n \neq \varnothing).$$

Proof. Assume that μ is σ-smooth and $F_n \downarrow \varphi$. The σ-smoothness $\Rightarrow \mu(\lim F_n) = \lim \mu(F_n)$, but these are unequal, contradicting $F_n \downarrow \varphi$. In general, let $G_n \in \mathscr{F}_\mu$. Let $F_1 = G_1$, $F_2 = G_1 \cap G_2$, etc. $\cap F_n = \cap G_n$. By above, $\cap F_n \neq \varnothing$, and $\mathscr{F}_\mu$ has C.I.P.

Conversely, assume that $\mathscr{F}_\mu$ has C.I.P. Suppose $A_{n_0} \in \mathscr{F}_\mu$. Than $A_n \notin \mathscr{F}_\mu$ for all $n \geq n_0$. $\mu(A_n) = 0$ for $n \geq n_0$ and μ is continuous from above at φ. By a standard result in measure theory, μ is σ-smooth.

The next results indicate the change in the theory when the measure is not assumed to be $\mathscr{L}$ regular.

DEFINITION. A filter is prime if $A \cup B \in F \Rightarrow A \in F$ or $B \in F$.

LEMMA 2.4. *Let μ be a 0 — 1 measure, not necessarily $\mathscr{L}$ regular. Let $\mathscr{F} = \{L \in \mathscr{L} \mid \mu(L) = 1\}$. Then $\mathscr{F}$ is a prime filter.*

LEMMA 2.5. *Let $\mathscr{F}$ be a prime $\mathscr{L}$ filter. For $A \in \mathscr{L}$ define $\mu(A) = 1$ if $A \in \mathscr{F}$; $\mu(A) = 0$ if $A \notin \mathscr{F}$. Then μ is a measure on $\mathscr{L}$ in the sense that $\mu(\varnothing) = 0$ and that μ is finitely additive.*

Proofs. The proofs of both these lemmas are straightforward calculations.

DEFINITION. A semi-ring P, is a non-empty collection of subsets which satisfy the following: *a*) closure under finite intersection; *b*) $E \in P$, $F \in P$, $F \supset E \Rightarrow$ there exists $\{C_i\}$, $C_i \in P$ such that $E = C_0 \subset C_1 \cdots \subset C_n = F$ and $D_i = C_i - C_{i-1} \in P$.

Let $\mathscr{L}$ be a lattice. Let $P(\mathscr{L}) = \{ F - E \mid F, E \in \mathscr{L}, F \supset E \}$. It is not difficult to show that $P(\mathscr{L})$ is a semiring. If we define for $D = F - E \in P(\mathscr{L})$ $\mu'(D) = \mu(F) - \mu(E)$, where μ is a 'measure' on the lattice, than μ' is a 'measure' on the semiring. It is always possible to extend such a measure from a semiring to an algebra. (See, for example, [7], p. 25–6). Using this result, and the previous lemmas, we have the following theorem, analogous to Theorem 2.1

THEOREM 2.3. *There is a 1 : 1 correspondence between* 0 — 1 *measures defined on* $\mathscr{A}(\mathscr{L})$, *for some* $\mathscr{L}$, *and prime* $\mathscr{L}$ *filters.*

Lastly, we will be discussing products of lattices, $\mathscr{L}_1 \times \mathscr{L}_2$, whose members do not form a lattice because they are not closed under unions. We now indicate how to associate with such a set a lattice, while preserving the measure-filter correspondence discussed above.

DEFINITION. A multiplicative system is a collection of sets, $\mathscr{M}$, satisfying the following: *a)* $\varnothing, X \in \mathscr{M}$; *b)* $\bigcap_1^N M_i \in \mathscr{M}$ whenever $M_i \in \mathscr{M}$ Let $\mathscr{L}(\mathscr{M}) = \left\{ \bigcup_1^n A_i \mid A_i \in \mathscr{M} \right\}$. We note that $\mathscr{L}(\mathscr{M})$ is a lattice $\supset \mathscr{M}$, and that $\mathscr{A}(\mathscr{L}(\mathscr{M})) = \mathscr{A}(\mathscr{M})$.

LEMMA 2.6. *Let* μ *be a* 0 — 1 *measure defined on* $\mathscr{A}(\mathscr{M}) = \mathscr{A}(\mathscr{L}(\mathscr{M}))$. *Then* μ *is* $\mathscr{M}$*-regular iff* μ *is* $\mathscr{L}(\mathscr{M})$*-regular.*

Proof. Suppose μ is $\mathscr{M}$-regular. Let $K \in \mathscr{A}(\mathscr{M})$. If $\mu(K) = 1$, than $\sup \mu(M) = 1$, $K \supset M$, $M \in \mathscr{M}$. But $\sup_{M \in \mathscr{M}} \mu(M) \leq \sup_{M \in \mathscr{L}(\mathscr{M})} \mu(M) \Rightarrow \sup_{M \in \mathscr{L}(\mathscr{M})} u(M) = 1$ The case when $\mu(K) = 0$ and the converse are proved similarly.

THEOREM 2.4. *There exists a 1 : 1 correspondence between prime filters on* $\mathscr{M}$, *prime filters on* $\mathscr{L}(\mathscr{M})$, *and* 0 — 1 *measures on* $\mathscr{A}(\mathscr{M}) = \mathscr{A}(\mathscr{L}(\mathscr{M}))$.

Proof. The correspondences are the following: Given G, a prime $\mathscr{M}$-filter, let $F = \{ B \in \mathscr{L}(\mathscr{M}) \mid B \supset A \in G \}$. F is directly shown to be a prime $\mathscr{L}(\mathscr{M})$ filter. By Theorem 2.3 this filter is in 1 : 1 correspondence with a 0 — 1 measure defined on $\mathscr{A}(\mathscr{L}(\mathscr{M}) = \mathscr{A}(\mathscr{M}))$.

Conversely, given μ, a 0 — 1 measure on $\mathscr{A}(\mathscr{M}) = \mathscr{A}(\mathscr{L}(\mathscr{M}))$, let $G = \{A \in \mathscr{M} \mid \mu(A) = 1\}$ G is easily seen to be a prime $\mathscr{M}$ filter. By Theorem 2.3 we have a correspondence between prime $\mathscr{L}(\mathscr{M})$ filters and 0 — 1 measures on $\mathscr{A}(\mathscr{L}(\mathscr{M}))$. The above construction gives us a correspondence between 0 — 1 measures on $\mathscr{A}(\mathscr{L}(\mathscr{M})) = \mathscr{A}(\mathscr{M})$ and prime $\mathscr{M}$ filters. This completes the proof.

THEOREM 2.5. *There is a 1 : 1 correspondence between ultrafilters on* $\mathscr{M}$, *ultrafilters on* $\mathscr{L}(\mathscr{M})$, 0 — 1 $\mathscr{M}$ *regular measures on* $\mathscr{A}(\mathscr{M})$ *and* 0 — 1 $\mathscr{L}(\mathscr{M})$ *regular measures on* $\mathscr{A}(\mathscr{L}(\mathscr{M}))$.

Proof. In the proof of Theorem 2.1 the closure under finite unions was never used. The result was actually more general: a 1 — 1 correspondence between 0 — 1 $\mathscr{M}$ regular measures and $\mathscr{M}$ ultrafilters, where $\mathscr{M}$ is a multiplicative system. By Lemma 2.6 there is a 1 : 1 correspondence between 0 — 1 $\mathscr{M}$ regular and 0 — 1 $\mathscr{L}(\mathscr{M})$ regular measures. We now get the correspondence between $\mathscr{M}$ ultrafilters and $\mathscr{L}(\mathscr{M})$ ultrafilters via the associated measures.

3. WALLMAN TOPOLOGIES AND VAGUE TOPOLOGIES

Given a lattice, $\mathscr{L}$, we will denote the space of $\mathscr{L}$-regular, 0 — 1 measures by $I_R(\mathscr{L})$. For $A \in \mathscr{L}$, let $W(A) = \{\mu \in I_R(\mathscr{L}) \mid \mu(A) = 1\}$. We define a topology on $I_R(\mathscr{L})$ by letting $\{W(A), A \in \mathscr{L}\}$ be a base for the closed sets. We will refer to this Wallman-type topology as the O_w topology.

THEOREM 3.1. *$(I_R(\mathscr{L}), O_w)$ is a T — 1 space.*

Proof. Assume that $\mu \neq \bar{\mu}$, both in $I_R(\mathscr{L})$. Then there exists $A \in \mathscr{L}$ such that $\mu(A) = 1$, $\bar{\mu}(A) = 0$. $\bar{\mu}(A') = 1 \Rightarrow A' \supset B \in \mathscr{L}$ and $\bar{\mu}(B) = 1$. $(W(A))'$ and $(W(B))'$ are the required neighborhoods.

THEOREM 3.2. *$(I_R(\mathscr{L}), O_w)$ is compact.*

Proof. We use the following characterization of compactness: A space X is compact iff every family of closed sets in X which has the finite intersection property has a non empty intersection. Let $H = \{W(A), A \in \mathscr{L}\}$ basic closed sets with F.I.P. Let $F = \{A \mid W(A) \in H\}$. Now $A, B \in F \Rightarrow$ $\Rightarrow W(A) \cap W(B) \neq \varphi \Rightarrow W(A \cap B) \neq \varphi \Rightarrow A \cap B \neq \varphi$, so F has the F.I.P. and can be extended to a filter, and to an ultrafilter G. Let μ_G be the measure associated with G. $W(A) \in H$ implies $A \in F$ which implies $A \in G$. So $\mu_G(A) = 1$, for all $A \in H$. But this implies that $\mu_G \in \cap W(A)$, and this implies that $I_R(\mathscr{L}), O_w$ is compact.

We would like to relate the O_w topology to the ' vague ' topology discussed by Alexandroff [1] and by Varadarajan [11].

Varadarajan starts with a topological space, X. His measures are not two valued and are regular with respect to zero sets-inverse images of zero under continuous functions. Calling this space of measures M (X), he defines a topology. For any measure, $m_0 \in M(X)$, consider sets $N(m_0; g_1, \cdots, g_n, \varepsilon) =$ $= \left\{ m : \left| \int_x g_r (dm - dm_0) \right| < \varepsilon ; r = 1, 2, \cdots, n ; g_1, \cdots, g_n \in C(X) \right\}$. The class of all such sets obtained by varying m_0, $\varepsilon > 0$ and g_i generates a system of neighborhoods for a topology. We will refer to this as the vague topology. (Varadarajan calls it the weak topology) A net $\{m_\alpha\}$ in M (X) converges to m in M (X) iff $\int_x g \, dm_\alpha \to \int_x g \, dm$ for all $g \in C(X)$.

A natural generalization of Varadarajan's framework, which the author has discussed in [8], is to consider $M_R(\mathscr{L})$, measures, not necessarily two-valued, but regular with respect to an arbitrary lattice $\mathscr{L}$. Continuous functions would be lattice-continuous functions, i.e., $g \in C_{\mathscr{L}}(X)$ if g^{-1} (an open set in R) $= L'$, where $L \in \mathscr{L}$. $I_R(\mathscr{L}) \subset M_R(\mathscr{L})$ and inherits the vague topology as a subset. What is the relationship between $(I_R(\mathscr{L}), O_w)$ and $(I_R(\mathscr{L})$, vague-inherited)? We prove the following:

THEOREM 3.3. $(I_R(\mathscr{L}), O_w)$ *and* $(I_R(\mathscr{L})$, *vague-inherited*) *are identical.* For the proof, we need a result of Alexandroff.

THEOREM. (Alexandroff). *In order that a sequence of non-negative measures should converge vaguely to* μ, *necessary and sufficient conditions are*:

a) $\lim_{m\to\infty} \mu_m(X) = \mu(X)$; *b*) *For every* $G = F'$, $F \in \mathscr{L}$, *either* $\mu(G) \leq \underline{\lim}\, \mu_m(G)$ *or* $\mu(F) \geq \overline{\lim}\, \mu_m(F)$.

Proof. A complete proof is in [1]. We note that Alexandroff did not speak in terms of lattice elements or complements but rather in terms of open and closed sets. His restriction to closed sets, however, was only that they satisfy the requirements for a lattice, and his result about open and closed sets is actually the more general result we have just given.

Proof. (of Theorem 3.3). We prove that the topologies are identical by proving that they have the same limit points. Assume that $\mu_m \to \mu$ in the O_w topology. If $F \in \mathscr{L}$, $\mu(F) = 0$ implies that $\mu \in (W(F))'$. Since $\mu_m \to \mu$, $m > N \Rightarrow \mu_m \in (W(F))'$ and $\mu_m(F) = 0$ for $m > N$. $\mu(F) = \lim \mu_m(F)$. If $\mu(F) = 1$, $\mu(F) \geq \overline{\lim}\, \mu_m(F)$. By Alexandroff's theorem, $\mu_m \to \mu$ in the vague topology.

Conversely, suppose that $\mu(G) \leq \underline{\lim}\, \mu_m(G)$ for all $G \in \mathscr{L}'$. A neighborhood of μ is $(W(K))'$. This implies that $\mu(K') = 1 \Rightarrow \underline{\lim}\, \mu_m(K') = 1 \Rightarrow \lim \mu_m(K') = 1$. If $m > N$, $\mu_m(K) = 0$ and μ_m is eventually in every neighborhood of μ, and $\mu_m \to \mu$ in the O_w topology.